A physicist's derivation of APS index theorem

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(With **Atish Dabholkar, Diksha Jain** - arXiv:**1905.05207**)

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Part I: Index theorem

(“Number” of eigen-spinors of Dirac operator with zero eigenvalue)

Let’s start from the basics of the Dirac operator

- euclidean, compact (without boundary), *spin* manifold

  can define Dirac fermion on the manifold

- *even-dimensional, orientable* manifold to define chirality

$$\gamma_{2n+1} = \frac{1}{(2n)!}\epsilon_{\mu_1\cdots\mu_{2n}}\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}$$
$$(\gamma_{2n+1})^2 = 1$$

  So the eigenvalue of γ_{2n+1} is ± 1. It is called chirality.
  On an even-dimensional, orientable manifold one can define chiral fermion on the manifold.

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- consider the Dirac eigenvalue problem

$$\not{D}\,\psi = \lambda\,\psi$$

$\not{D}$ is the Dirac operator in presence of gauge and/or spin connection

- *Define* Chiral spinors

$$\psi_\pm = \mathcal{P}_\pm \psi \quad , \quad \mathcal{P}_\pm = \left(\frac{1 \mp \gamma_{2n+1}}{2}\right)$$

- action of Dirac operator flips chirality

$$\not{D}\,\mathcal{P}_\pm = \mathcal{P}_\mp\,\not{D} \quad \longleftrightarrow \quad \not{D}\psi_\pm = \lambda\,\psi_\mp$$

- for every eigenspinor of non-zero eigenvalue and positive chirality, there exists an eigenspinor of same eigenvalue and negative chirality.

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the number of positive (negative) chirality zero mode n_+ (n_-).

what is the value of the following quantity (Dirac index) ?

$$\text{index}\ (\not{D}) = n_+ - n_-$$

Atiyah-Singer index theorem

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Consider an even-dimensional, orientable, compact manifold (with no boundary) $\mathcal{M}$. Let the metric on $\mathcal{M}$ be $g_{\mu\nu}$. And the Riemann tensor constructed out of metric is

$$R^{ab}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$$

Then the Dirac index in $D = 4$, is given by

$$-\frac{1}{24}\int_{\mathcal{M}} \frac{\mathrm{tr}\, R \wedge R}{16\pi^2}$$

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Application of index theorem

- Anomaly is (controlled) quantum violation of a classical symmetry
- Massless Dirac fermion enjoys the following symmetry (Chiral symmetry)

$$\Psi(x) \longrightarrow \exp\left[\mathbf{i}\, \gamma_{2n+1}\, \theta\right] \Psi(x)$$

- However, the path integral measure doesn't obey this symmetry. Non-invariance of the measure comes only from the zero mode of the Dirac operator and hence controlled by the index theorem.

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Derivation using supersymmetry

- A quantum mechanical system is defined a Hamiltonian

$$H|\Psi\rangle = E|\Psi\rangle$$

- A Supersymmetric QM system is actually a pair* of Quantum mechanical system.

$$|b\rangle \qquad , \qquad |f\rangle$$

It is defined by a super-charge

$$Q^2 = H \qquad , \qquad Q|b\rangle = |f\rangle \qquad , \qquad Q|f\rangle = |b\rangle$$

- For supersymmetric systems, we can define Witten index which roughly counts the number of bosons minus the number of fermions

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A path integral derivation (Witten, Alvarez-Gaume, Friedan, Windey)

- Consider super-symmetric quantum mechanics with one real super-charge whose (bosonic) target space is the compact manifold

$$\frac{1}{2}\int dt\left[g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}+\mathbf{i}\,\delta_{ab}\,\psi^a\left(\frac{d\psi^b}{dt}+\omega_{akb}\frac{dx^k}{dt}\psi^b\right)\right]$$

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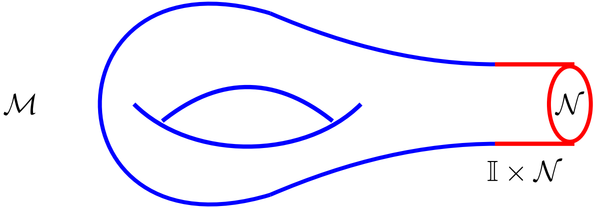
Relation between spacetime variable and world-line variables

$$\begin{aligned} \not{D} &\longleftrightarrow \mathcal{Q} \\ \not{D}^2 &\longleftrightarrow H \\ \gamma^{2n+1} &\longleftrightarrow (-1)^F \\ \mathrm{index}(\not{D}) &\longleftrightarrow W(\infty) = W(0) \end{aligned}$$

where

$$W(\beta) = \mathrm{Tr}_{\mathcal{H}}\,(-1)^F e^{-\beta H}$$

Dirac index for manifold with boundary



Atiyah-Patodi-Singer index theorem

Variation problem of the Dirac action

The **boundary term** for the variation of the Dirac action is (roughly) of the form

$$\int_{\partial\mathcal{M}} \left[\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-\right]$$

One can impose (local) boundary condition

$$\psi_+\Big|_{\partial\mathcal{M}} = \pm\psi_-\Big|_{\partial\mathcal{M}}$$

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Boundary condition I

- Any local boundary doesn't preserve chiral current and hence not good for index problem
- APS invented a non-local boundary condition to define the index problem
- However, this boundary condition can be thought of as a euclidean continuation of Feynman $i\varepsilon$ (scattering) boundary condition.

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Boundary condition II

- For a particular choice of the gamma matrices, the Dirac operator near the boundary can be written as

$$\begin{bmatrix} 0 & \partial_u + \mathcal{B} \\ -\partial_u + \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix} = \sqrt{E} \begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix}$$

- (say) We diagonalize the boundary operator $\mathcal{B}$

$$\mathcal{B}\,\chi_\lambda = \lambda \chi_\lambda$$

- Focus on zero modes ($E = 0$). Locally near the boundary

$$\Psi_\pm(u) = \sum_\lambda \exp\left[\mp\lambda\, u\right] \chi_\lambda$$

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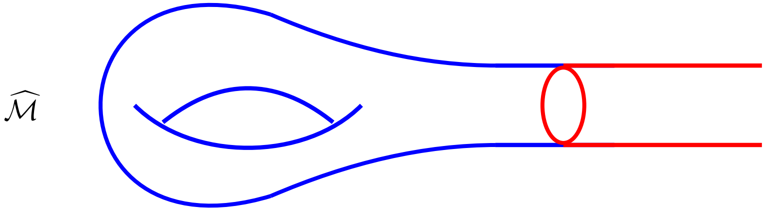

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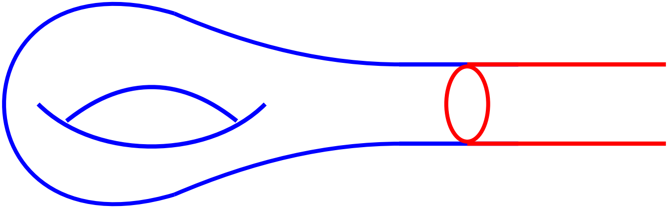

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η invariant

Given an operator $\mathcal{B}$ and its eigenvalues

$$\mathcal{B}\,\chi_\lambda = \lambda\,\chi_\lambda \qquad , \qquad \lambda \in \mathbb{R}$$

We can define the following quantity which defines the spectral asymmetry ($\lambda \neq 0$)

$$\eta = \sum_\lambda \mathrm{sgn}(\lambda)$$

Need to introduce a regulator

$$\begin{aligned} \eta_{\mathrm{APS}}(s) &= \sum_\lambda \frac{\lambda}{|\lambda|^{s+1}} = \sum_\lambda \frac{\mathrm{sgn}(\lambda)}{|\lambda|^s} \\ \eta_{\mathrm{PI}}(\beta) &= \sum_\lambda \mathrm{sgn}(\lambda)\,\mathrm{erfc}\left(|\lambda|\sqrt{\beta}\right) \end{aligned}$$

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η is **not** a topological quantity - can change under deformation.

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2. fermion fractionzation - charge of the vacuum in presence of soliton
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APS index theorem

$$\mathrm{index}(\not{D}) = \int_{\mathcal{M}} \alpha(x) - \frac{1}{2}\eta$$

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Revisiting the basics

- Let's start from the basic defn of Witten index

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- For compact manifold (with/without boundary), the spectrum is discrete. One can use supersymmetry to prove $W(\beta)$ gets contribution only from zero energy states and hence

$$W(\beta) = W(0) = W(\infty)$$

- Index of an operator is defined as

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Non-compact extension

We add the trivial cylinder. This does the following thing

- introduces continuum of scattering states
- The states of the compact manifold is simply the bound states of the non-compact manifold
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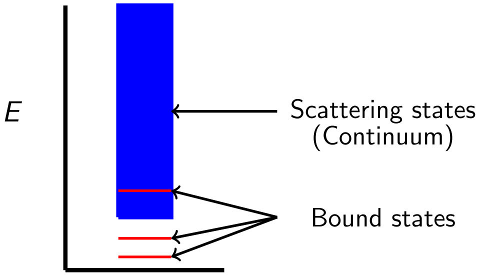
Assumption: we *assume* that the continuum is separated from zero



$$\widehat{W}(\beta) = \mathrm{Tr}_{bound}\left[(-1)^{\mathrm{F}} e^{-\beta H}\right] + \mathrm{Tr}_{scattering}\left[(-1)^{\mathrm{F}} e^{-\beta H}\right]$$

Since we assumed that the scattering state is separated from zero,

$$\widehat{W}(\infty) = \lim_{\beta \to \infty} \mathrm{Tr}_{bound}\left[(-1)^{\mathrm{F}} e^{-\beta H}\right]$$

So this gives the index of compact manifold. However it is more difficult to compute. So rewrite the above equation as

$$\begin{aligned} \widehat{W}(\infty) &= \widehat{W}(0) + \left[\widehat{W}(\infty) - \widehat{W}(0)\right] \\ &\simeq \mathrm{AS} - \frac{1}{2}\eta \end{aligned}$$

Computing η invariant

We start from our guess

$$\begin{aligned}\eta(\beta) &:= 2(\widehat{W}(\beta) - \widehat{W}(\infty)) \\ &= 2\sum_{\lambda}\int dk\left[\rho_{+}^{\lambda}(k) - \rho_{-}^{\lambda}(k)\right]e^{-\beta E(k)}\end{aligned}$$

Now the difference of density of state is related to deference of phase shift

$$\rho_{+}^{\lambda}(k) - \rho_{-}^{\lambda}(k) = \frac{1}{\pi}\frac{d}{dk}\left[\delta_{+}^{\lambda}(k) - \delta_{-}^{\lambda}(k)\right] .$$

$\delta_{\pm}^{\lambda}(k)$ are the phase shifts.

Let the asymptotic form of the scattering wave functions is

$$\psi_{\pm}^{\lambda\,k}(u) \quad \sim \quad c_{\pm}^{\lambda}\left[e^{\mathbf{i}ku} + e^{\mathbf{i}\delta_{\pm}^{\lambda}(k)-\mathbf{i}ku}\right]$$

where $\delta_{\pm}^{\lambda}(k)$ are the phase shifts.
Now one can use supersymmetry to determine the difference of phase shift just from the asymptotic data

$$2\delta_{+}^{\lambda}(k) - 2\delta_{-}^{\lambda}(k) = -\mathbf{i}\ \ln\left(\frac{\mathbf{i}k+\lambda}{\mathbf{i}k-\lambda}\right) + \pi$$

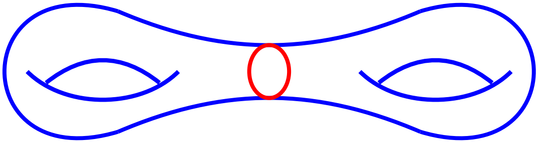
So the final result is

$$\sum_{\lambda} \mathrm{sgn}(\lambda)\mathrm{Erfc}\left(|\lambda|\sqrt{\frac{\beta}{2}}\right)$$

Computing AS piece

Double it to get a compact manifold without boundary

$\overline{\mathcal{M}}$



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Part II: Mock modular form

Modular form

Modular group $SL(2,\mathbb{Z})$:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \quad , \qquad ad - bc = 1 \qquad a,b,c,d \in \mathbb{Z}$$

Modular form $f(\tau)$:

- Holomoprhic on the upper half plane $\mathrm{Im}\tau > 0$
- Obeys the following relation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

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Mock-modular form (Ramanujan, Zwegers, Dabholkar-Murthy-Zagier, ...)

Consider a pair h, g such that

- h is holomorphic but not modular
- $\hat{h} = h + \bar{g}$ is modular but not holomorphic
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Three clues

- The elliptic genus of a compact (without boundary) CFT can be written as a sum of Dirac indices.

- The elliptic genus of a non-compact CFT is (mixed-)mock modular.

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Is there a connection ?

Let's compare two formulae

$$\mathrm{index}(\not{D}) = \int_{\mathcal{M}} \alpha(x) - \frac{1}{2}\eta$$

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partition functions - counts supersymmetric states

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Only these pieces are there for compact manifold without boundary; they are β $(\bar{\tau})$ independent

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Sensitive only to the boundary

Don't know in general !

However, we observe some connections in case of cigar !

Cigar

Cigar is a two dimensional non-compact manifold. The metric is given by

$$ds^2 = k\,(d\rho^2 + \tanh^2\rho\, d\psi^2)$$

ψ is a periodic direction with period 2π. The Dirac operator near the boundary takes the form

$$\begin{aligned} \mathbf{i}\not{D} &= \gamma^r(\mathbf{i}\partial_r - w\,K_r) + \gamma^\theta(\mathbf{i}\partial_\theta - w\,K_\theta) \\ &= i\gamma^r\left[\partial_r - \frac{1}{\tanh r}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(\mathbf{i}\partial_\theta - w\,k\tanh^2 r)\right] \end{aligned}$$

Lots of work on Cigar Elliptic genus by Various people
Ashok, Doroud, Troost; Murthy,...

We computed from supersymmetric sigma model.

Observations

The EG of Cigar SCFT is a Mock Jacobi form of weight 1/2.

$$-i\frac{\vartheta_1(\tau,z)}{\eta^3(\tau)}\sum_w\sum_n\left[\frac{1}{2}\mathrm{sgn}\left(\frac{n}{k}-w\right)\mathrm{Erfc}\left(\sqrt{k\pi\tau_2}\left|w-\frac{n}{k}\right|\right)\right.$$

$$\left.-\mathrm{sgn}(n\,\mathrm{im}\ \tau)\,\Theta\left[w\left(\frac{n}{k}-w\right)\right]\right]q^{-(n-wk)^2/4k}q^{(n+wk)^2/4k}y^{J_L}$$

Observations

In the $\tau_2 \to \infty$ limit we obtain

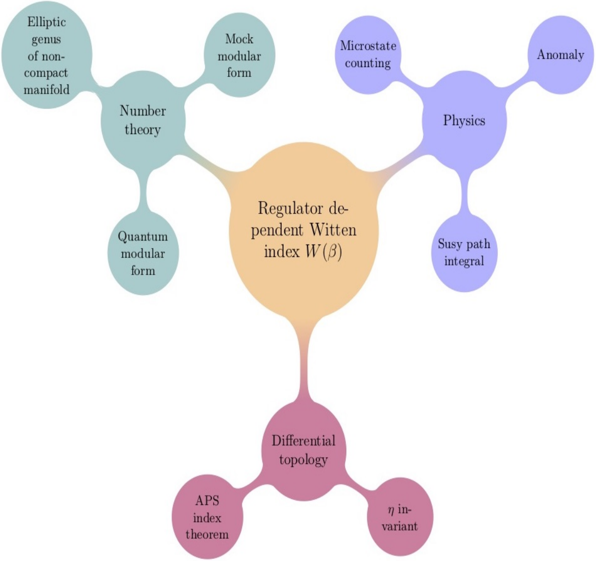
$$\widehat{\chi}(\tau, \bar{\tau}|z) = 0$$

In the $\tau_2 \to 0$ limit we obtain

$$-i\frac{\theta_1(\tau_1, z)}{\eta(\tau_1)^3} \sum_{w,n} \left[(-\mathrm{sgn}(n) + \frac{1}{2}\mathrm{sgn}\left(\frac{n}{k} - w\right) \right] e^{2\pi i \tau_1 n w} y^{\frac{n+wk}{k}}$$

But we know the AS piece vanishes in 2 dimensions
One can consider the radial limit ($\tau_2 \to 0^+$) of the non-holomophic part to obtain (vector-valued) 'quantum modular forms' (in this case the weight is $1/2$)

Summary



Regulator dependent Witten index $W(\beta)$
Number theory
Elliptic genus of non-compact manifold
Mock modular form
Quantum modular form
Physics
Microstate counting
Anomaly
Susy path integral
Differential topology
APS index theorem
η invariant

Future directions

1. To Extend the proof APS index theorem using scattering for a more general type of manifolds
2. Is it possible to make the connection non-compact elliptic genus and APS index precise
3. Given a Dirac operator on a manifold with boundary, we know how to compute and boundary operator and it's eta invariant. Similarly, given a non-compact CFT, is it possible to write down systematic steps to compute the shadow?
4. ...

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Thank you