Holographic Form for Wilson's RG

B. Sathiapalan

Matscience, Chennai

IIT Bombay 24-9-2019

Outline

Holography

This talk is about an attempt to understand AdS/CFT or Holography in terms of Wilsonian RG.
Based on arXiv:1706.03371 (Nucl.Phys. B) and arXiv:1902.02486(Nucl. Phys. B) with Hidenori Sonoda (Kobe University).

Holography

- Holography is a conjectured correspondence between a $d + 1$-dimensional (gravity) bulk theory and a d-dimensional boundary field theory.
- Lot of evidence by now.
- Main Feature (for our purposes): The extra dimension of the bulk corresponds to the scale of the boundary field theory.

Holography

- Holography is a conjectured correspondence between a $d + 1$-dimensional (gravity) bulk theory and a d-dimensional boundary field theory.
- Lot of evidence by now.
- Main Feature (for our purposes): The extra dimension of the bulk corresponds to the scale of the boundary field theory.

Holography

- Holography is a conjectured correspondence between a $d+1$-dimensional (gravity) bulk theory and a d-dimensional boundary field theory.
- Lot of evidence by now.
- Main Feature (for our purposes): The extra dimension of the bulk corresponds to the scale of the boundary field theory.

RG

- If the holographic conjecture is right then evolution in this extra direction SHOULD correspond to RG evolution in the field theory.

- Central Theme of this talk: Can we understand (or even better, derive) this correspondence starting from Wilsonian RG i.e. without assuming this conjecture ?

RG

- If the holographic conjecture is right then evolution in this extra direction SHOULD correspond to RG evolution in the field theory.

- Central Theme of this talk: Can we understand (or even better, derive) this correspondence starting from Wilsonian RG i.e. without assuming this conjecture ?

AdS/CFT Conjecture- Maldacena 1997

- Duality between boundary conformal field theory in d-dimensional flat space and bulk gravity theory in AdS_{d+1}
- AdS metric is
$$ds^2 = \frac{dz^2 + dx^i dx^i}{z^2}$$
Boundary is placed at $z = \epsilon$.
- Radial coordinate z defines the length scale of the boundary.

AdS/CFT Conjecture- Maldacena 1997

- Duality between boundary conformal field theory in d-dimensional flat space and bulk gravity theory in AdS_{d+1}
- AdS metric is

$$ds^2 = \frac{dz^2 + dx^i dx^i}{z^2}$$

Boundary is placed at $z = \epsilon$.

- Radial coordinate z defines the length scale of the boundary.

AdS/CFT Conjecture- Maldacena 1997

- Duality between boundary conformal field theory in d-dimensional flat space and bulk gravity theory in AdS_{d+1}
- AdS metric is

$$ds^2 = \frac{dz^2 + dx^i dx^i}{z^2}$$

Boundary is placed at $z = \epsilon$.

- Radial coordinate z defines the length scale of the boundary.

AdS/CFT Conjecture- Maldacena 1997

- Duality between boundary conformal field theory in d-dimensional flat space and bulk gravity theory in AdS_{d+1}
- AdS metric is

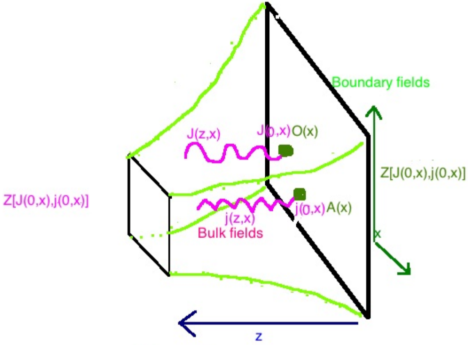
$$ds^2 = \frac{dz^2 + dx^i dx^i}{z^2}$$

  Boundary is placed at $z = \epsilon$.
- Radial coordinate z defines the length scale of the boundary.

Mathematical Statement of Duality

- Equality of generating functionals: (x - coordinates of boundary and z is radial coordinate of AdS)
-
$$Z[J_0]_{bulk} = \int_{J(x,\epsilon)=J_0(x)} \mathcal{D}\, \underbrace{J(x,z)}_{bulk\ \ field}\, e^{-S_{gravity}[J(x,z)]}$$

$$= \int \underbrace{\mathcal{D}\, A(x)}_{boundary\ \ field}\; e^{-S_{field\ \ theory}[A]+\int_{z=\epsilon} dx\ J_0(x) O[A]} = Z[J_0]_{boundary}$$



Holography : Z[J(0,x),j(0,x)] = Z[J(0,x),j(0,x)]

AdS Action

- The AdS action for a scalar field

$$S_{gravity}[J] =$$

$$\int_{z=\epsilon}^{z=\infty} dz \int d^d x \, z^{-d+1} [(\partial_z J \partial_z J + \partial_i J \partial_i J) + \frac{1}{z^2} m^2 J^2 + \ldots$$

- Semiclassical Prescription: Solve the EOM with boundary conditions at $z = \epsilon$ and $z = \infty$ and evaluate (on-shell) action to get Green function $G(p, \epsilon)$ of the boundary theory:

$$Z[J_0] = e^{-\frac{1}{2}\int_p J_0(p) J_0(-p) G(p,\epsilon)}$$

.

AdS Action

- The AdS action for a scalar field

$$S_{gravity}[J] =$$

$$\int_{z=\epsilon}^{z=\infty} dz \int d^d x \, z^{-d+1}[(\partial_z J \partial_z J + \partial_i J \partial_i J) + \frac{1}{z^2} m^2 J^2 + ...$$

- Semiclassical Prescription: Solve the EOM with boundary conditions at $z = \epsilon$ and $z = \infty$ and evaluate (on-shell) action to get Green function $G(p, \epsilon)$ of the boundary theory:

$$Z[J_0] = e^{-\frac{1}{2}\int_p J_0(p) J_0(-p) G(p,\epsilon)}$$

.

Holographic RG

- ϵ plays the role of RG scale
- Evolution in ϵ is by EOM of bulk theory. This is like an RG: "Holographic RG"

Holographic RG

- ϵ plays the role of RG scale
- Evolution in ϵ is by EOM of bulk theory. This is like an RG: "Holographic RG"

Wilsonian RG

- What we do from now on is logically independent of AdS/CFT : We will not use the AdS/CFT conjecture - we will use Wilsonian Exact RG.
- Two steps:
- 1. Obtain a functional integral representation of Wilson's ERG evolution operator.
- 2. Change variables ("coarse grain with anomalous dimensions") to obtain an AdS action.

Wilsonian RG

- What we do from now on is logically independent of AdS/CFT : We will not use the AdS/CFT conjecture - we will use Wilsonian Exact RG.
- Two steps:
- 1. Obtain a functional integral representation of Wilson's ERG evolution operator.
- 2. Change variables ("coarse grain with anomalous dimensions") to obtain an AdS action.

Wilsonian RG

- What we do from now on is logically independent of AdS/CFT : We will not use the AdS/CFT conjecture - we will use Wilsonian Exact RG.
- Two steps:
- 1. Obtain a functional integral representation of Wilson's ERG evolution operator.
- 2. Change variables ("coarse grain with anomalous dimensions") to obtain an AdS action.

Wilsonian RG

- What we do from now on is logically independent of AdS/CFT : We will not use the AdS/CFT conjecture - we will use Wilsonian Exact RG.
- Two steps:
- 1. Obtain a functional integral representation of Wilson's ERG evolution operator.
- 2. Change variables ("coarse grain with anomalous dimensions") to obtain an AdS action.

Polchinski's Equation

- Step1: Consider a zero dimensional field variable x (x is going to become a field variable in d dimensions eventually). Let the "Euclidean action" be:

$$S = \underbrace{\frac{1}{2} G^{-1}(t) x^2}_{Kinetic} + \underbrace{S_I(x)}_{Interaction}$$

G is a propagator and t will later be identified with $\ln \frac{\Lambda_0}{\Lambda}$.

- Polchinski's Eqn

$$\frac{\partial S_I}{\partial t} = -\frac{1}{2} \dot{G}(t) [\frac{\partial^2 S_I}{\partial x^2} - (\frac{\partial S_I}{\partial x})^2]$$

Condition for partition function to be invariant as t increases.

Polchinski's Equation

- Step1: Consider a zero dimensional field variable x (x is going to become a field variable in d dimensions eventually). Let the "Euclidean action" be:

$$S = \underbrace{\frac{1}{2}G^{-1}(t)x^2}_{Kinetic} + \underbrace{S_I(x)}_{Interaction}$$

G is a propagator and t will later be identified with $\ln \frac{\Lambda_0}{\Lambda}$.

- Polchinski's Eqn

$$\frac{\partial S_I}{\partial t} = -\frac{1}{2}\dot{G}(t)[\frac{\partial^2 S_I}{\partial x^2} - (\frac{\partial S_I}{\partial x})^2]$$

Condition for partition function to be invariant as t increases.

Polchinski's Equation

- Step1: Consider a zero dimensional field variable x (x is going to become a field variable in d dimensions eventually). Let the "Euclidean action" be:

$$S = \underbrace{\frac{1}{2}G^{-1}(t)x^2}_{Kinetic} + \underbrace{S_I(x)}_{Interaction}$$

G is a propagator and t will later be identified with $\ln \frac{\Lambda_0}{\Lambda}$.

- Polchinski's Eqn

$$\frac{\partial S_I}{\partial t} = -\frac{1}{2}\dot{G}(t)[\frac{\partial^2 S_I}{\partial x^2} - (\frac{\partial S_I}{\partial x})^2]$$

Condition for partition function to be invariant as t increases.

Background: Why this equation?

ANS.: There is a variant of Wilson's eqn (1974)

$$\frac{\partial \psi(x,t)}{\partial t} = -\frac{1}{2}\dot{G}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2G^{-1}x\right)\psi(x,t)\right.$$

- If $\psi(x) = e^{-S}$, then

$$\psi(y,T) = e^{-\frac{1}{2}y^2} \int dx\, e^{\frac{1}{2}\frac{(y\sqrt{G(T)}-x)^2}{G(T)-G(0)}} e^{\frac{1}{2}G(0)^{-1}x^2}\psi(x,0)$$

- As $T \to 0$, $\psi(y,T) \to \psi(y,0)$ and as $T \to \infty$, $G(T) \to 0$, $\psi(y,T) \to e^{-\frac{1}{2}y^2} \int dx\, \psi(x,0)$. So at $T \to \infty$ we have completely integrated over the variable x - coarse graining. All information about the starting wave function is lost. For finite T, partial coarse graining.
- This (variant of) Wilson's equation is for S, Polchinski's is for S_I.

- If $\psi(x) = e^{-S}$, then

$$\psi(y,T) = e^{-\frac{1}{2}y^2} \int dx \; e^{\frac{1}{2}\frac{(y\sqrt{G(T)}-x)^2}{G(T)-G(0)}} \, e^{\frac{1}{2}G(0)^{-1}x^2} \psi(x,0)$$

- As $T \to 0$, $\psi(y,T) \to \psi(y,0)$ and as $T \to \infty$, $G(T) \to 0$, $\psi(y,T) \to e^{-\frac{1}{2}y^2} \int dx \; \psi(x,0)$. So at $T \to \infty$ we have completely integrated over the variable x - coarse graining. All information about the starting wave function is lost. For finite T, partial coarse graining.
- This (variant of) Wilson's equation is for S, Polchinski's is for S_I.

- If $\psi(x) = e^{-S}$, then

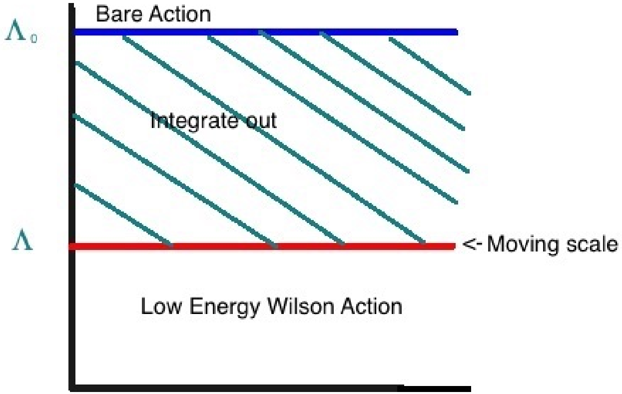
$$\psi(y,T) = e^{-\frac{1}{2}y^2} \int dx \; e^{\frac{1}{2}\frac{(y\sqrt{G(T)}-x)^2}{G(T)-G(0)}} e^{\frac{1}{2}G(0)^{-1}x^2} \psi(x,0)$$

- As $T \to 0$, $\psi(y,T) \to \psi(y,0)$ and as $T \to \infty$, $G(T) \to 0$, $\psi(y,T) \to e^{-\frac{1}{2}y^2} \int dx \; \psi(x,0)$. So at $T \to \infty$ we have completely integrated over the variable x - coarse graining. All information about the starting wave function is lost. For finite T, partial coarse graining.
- This (variant of) Wilson's equation is for S, Polchinski's is for S_I.

- If $\psi(x) = e^{-S}$, then

$$\psi(y,T) = e^{-\frac{1}{2}y^2} \int dx\, e^{\frac{1}{2}\frac{(y\sqrt{G(T)}-x)^2}{G(T)-G(0)}} e^{\frac{1}{2}G(0)^{-1}x^2} \psi(x,0)$$

- As $T \to 0$, $\psi(y,T) \to \psi(y,0)$ and as $T \to \infty$, $G(T) \to 0$, $\psi(y,T) \to e^{-\frac{1}{2}y^2} \int dx\, \psi(x,0)$. So at $T \to \infty$ we have completely integrated over the variable x - coarse graining. All information about the starting wave function is lost. For finite T, partial coarse graining.
- This (variant of) Wilson's equation is for S, Polchinski's is for S_I.


Bare Action
Λ_0
Integrate out
Λ
<- Moving scale
Low Energy Wilson Action

Diffusion equation

- Can be written as a linear equation in terms of $\psi' = e^{-S_I}$:

$$\frac{\partial \psi'}{\partial t} = -\frac{1}{2}\dot{G}(t)\frac{\partial^2 \psi'}{\partial x^2}$$

$\Longrightarrow$ Diffusion equation.

- The evolution operator is clearly

$$e^{-\frac{1}{2}\int_0^T dt\ \dot{G}\frac{\partial^2}{\partial x^2}} = e^{-\frac{1}{2}(G(T)-G(0))\frac{\partial^2}{\partial x^2}} = e^{\frac{1}{2}F(T)\frac{\partial^2}{\partial x^2}}$$

Here $F(T) = G(0) - G(T)$.

Diffusion equation

- Can be written as a linear equation in terms of $\psi' = e^{-S_I}$:

$$\frac{\partial \psi'}{\partial t} = -\frac{1}{2}\dot{G}(t)\frac{\partial^2 \psi'}{\partial x^2}$$

$\Longrightarrow$ Diffusion equation.

- The evolution operator is clearly

$$e^{-\frac{1}{2}\int_0^T dt\ \dot{G}\frac{\partial^2}{\partial x^2}} = e^{-\frac{1}{2}(G(T)-G(0))\frac{\partial^2}{\partial x^2}} \equiv e^{\frac{1}{2}F(T)\frac{\partial^2}{\partial x^2}}$$

Here $F(T) = G(0) - G(T)$.

Integral Kernel Representation

- For later use, another ways of writing this:

$$\psi(x_f, t_f) = \int dx_i \ e^{\frac{1}{2}\frac{(x_f - x_i)^2}{G_f - G_i}} \psi(x_i, t_i)$$

$G(T) = G_f$ and $G(0) = G_i$.

Functional Integral

- Thus the path integral representation is obvious:

$$\psi'(x',T) = \int dx \int_{x(0)=x;x(T)=x'} \mathcal{D}x(t)\ e^{\frac{1}{2}\int_0^T dt\ \frac{1}{G}(\dot{x})^2}\psi'(x,0)$$

- If we replace x by $x(p)$ and $G(t)$ to $G(p,t)$ we immediately generalize to higher dimensions.

Functional Integral

- Thus the path integral representation is obvious:

$$\psi'(x', T) = \int dx \int_{x(0)=x; x(T)=x'} \mathcal{D}x(t)\ e^{\frac{1}{2}\int_0^T dt\ \frac{1}{G}(\dot{x})^2} \psi'(x, 0)$$

- If we replace x by $x(p)$ and $G(t)$ to $G(p, t)$ we immediately generalize to higher dimensions.

Holographic Form

- In field theories, there is a cutoff function $K(p/\Lambda)$ (for e.g. $e^{\frac{-p^2}{\Lambda^2}}$)

$$F(p,T) = G(p,0) - G(p,T) = \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2} = \Delta_h$$

Δ_h is called the "high energy" propagator.

- CONCLUSION: We have a functional form for Polchinski's ERG eqn. The action is $d+1$-dimensional, where d is the dimension of the field theory we started with. $\Longrightarrow$ Holographic.
END OF STEP 1

Holographic Form

- In field theories, there is a cutoff function $K(p/\Lambda)$ (for e.g. $e^{\frac{-p^2}{\Lambda^2}}$)

$$F(p,T) = G(p,0) - G(p,T) = \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2} = \Delta_h$$

  Δ_h is called the “high energy" propagator.

- CONCLUSION: We have a functional form for Polchinski's ERG eqn. The action is $d+1$-dimensional, where d is the dimension of the field theory we started with. $\Longrightarrow$ Holographic.
  END OF STEP 1

Holographic Form

- In field theories, there is a cutoff function $K(p/\Lambda)$ (for e.g. $e^{\frac{-p^2}{\Lambda^2}}$)

$$F(p,T) = G(p,0) - G(p,T) = \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2} = \Delta_h$$

Δ_h is called the "high energy" propagator.

- CONCLUSION: We have a functional form for Polchinski's ERG eqn. The action is $d+1$-dimensional, where d is the dimension of the field theory we started with. $\Longrightarrow$ Holographic.

END OF STEP 1

Holographic Form

- In field theories, there is a cutoff function $K(p/\Lambda)$ (for e.g. $e^{\frac{-p^2}{\Lambda^2}}$)

$$F(p,T) = G(p,0) - G(p,T) = \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2} = \Delta_h$$

  Δ_h is called the “high energy" propagator.

- CONCLUSION: We have a functional form for Polchinski's ERG eqn. The action is $d+1$-dimensional, where d is the dimension of the field theory we started with. $\implies$ Holographic.
  END OF STEP 1

Non-standard action

- Step 2: the p dependence in the action is not the standard one - $(p^2 + m^2)$.
- Need change of variables

Non-standard action

- Step 2: the p dependence in the action is not the standard one - $(p^2 + m^2)$.
- Need change of variables

AdS_1

- Let $x = yf$ where $f^2 = -\dot{G}$. y is our new variable.
- Let us choose f to satisfy ($z = e^t$):

$$(z\frac{d}{dz})^2 e^{-\ln f} = (z^2 p^2 + m^2) e^{-\ln f}$$

- Then the action for y becomes:

$$\int \frac{dz}{z} [z^2(\frac{dy}{dz})^2 + y^2(z^2p^2 + m^2)]$$

Scalar field in AdS_1!

AdS_1

- Let $x = yf$ where $f^2 = -\dot{G}$. y is our new variable.

- Let us choose f to satisfy ($z = e^t$):

$$(z\frac{d}{dz})^2 e^{-\ln f} = (z^2p^2 + m^2)e^{-\ln f}$$

- Then the action for y becomes:

$$\int \frac{dz}{z}[z^2(\frac{dy}{dz})^2 + y^2(z^2p^2 + m^2)]$$

Scalar field in AdS_1!

AdS_1

- Let $x = yf$ where $f^2 = -\dot{G}$. y is our new variable.

- Let us choose f to satisfy ($z = e^t$):

$$(z\frac{d}{dz})^2 e^{-\ln f} = (z^2p^2 + m^2)e^{-\ln f}$$

- Then the action for y becomes:

$$\int \frac{dz}{z} [z^2(\frac{dy}{dz})^2 + y^2(z^2p^2 + m^2)]$$

Scalar field in AdS_1 !

AdS_1

- Let $x = yf$ where $f^2 = -\dot{G}$. y is our new variable.
- Let us choose f to satisfy ($z = e^t$):

$$(z\frac{d}{dz})^2 e^{-\ln f} = (z^2p^2 + m^2)e^{-\ln f}$$

- Then the action for y becomes:

$$\int \frac{dz}{z}\,[z^2(\frac{dy}{dz})^2 + y^2(z^2p^2 + m^2)]$$

Scalar field in AdS_1!

AdS_1

- Equation for $\frac{1}{f}$ is the same as the equation for y

$$[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - (p^2 + \frac{m^2}{z^2})]\frac{1}{f} = 0$$

  Solutions are Bessel functions -$K_m(pz)$, $I_m(pz)$.
- $f^2 = -\dot{G} \implies$ Constraints on G.
- All this goes through for any dimension, d, where we get a scalar field in AdS_{d+1}.

AdS_1

- Equation for $\frac{1}{f}$ is the same as the equation for y

$$[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - (p^2 + \frac{m^2}{z^2})]\frac{1}{f} = 0$$

Solutions are Bessel functions -$K_m(pz)$, $I_m(pz)$.

- $f^2 = -\dot{G} \implies$ Constraints on G.
- All this goes through for any dimension, d, where we get a scalar field in AdS_{d+1}.

AdS_1

- Equation for $\frac{1}{f}$ is the same as the equation for y

$$[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - (p^2 + \frac{m^2}{z^2})]\frac{1}{f} = 0$$

Solutions are Bessel functions -$K_m(pz)$, $I_m(pz)$.

- $f^2 = -\dot{G} \implies$ Constraints on G.
- All this goes through for any dimension, d, where we get a scalar field in AdS_{d+1}.

Higher dimensions

- Multiply and divide by z^d:

$$S_B = -\int dz z^{-d+1} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \frac{\partial x_p}{\partial z} \frac{\partial x_{-p}}{\partial z} \frac{1}{z^{-d} \dot{G}(p)}$$

- Let $z^{-d}\dot{G} = -f^2$ and require as before:

$$z^{d-1}(z^{-d+1}\frac{d}{dz})^2 e^{-\ln f} = z^{-d+1}(p^2 + \frac{m^2}{z^2})e^{-\ln f}$$

- Performing the same manipulations as before we get:

$$S_B = \int dz \int_p \{z^{-d+1}(\frac{\partial y_p}{\partial z}\frac{\partial y_{-p}}{\partial z}) + z^{-d+1}(p^2 + \frac{m^2}{z^2})y_p y_{-p}\}$$

Scalar field in AdS_{d+1}! END OF STEP 2

Higher dimensions

- Multiply and divide by z^d:

$$S_B = -\int dz z^{-d+1} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \frac{\partial x_p}{\partial z} \frac{\partial x_{-p}}{\partial z} \frac{1}{z^{-d}\dot{G}(p)}$$

- Let $z^{-d}\dot{G} = -f^2$ and require as before:

$$z^{d-1}(z^{-d+1}\frac{d}{dz})^2 e^{-\ln f} = z^{-d+1}(p^2 + \frac{m^2}{z^2})e^{-\ln f}$$

- Performing the same manipulations as before we get:

$$S_B = \int dz \int_p \{z^{-d+1}(\frac{\partial y_p}{\partial z}\frac{\partial y_{-p}}{\partial z}) + z^{-d+1}(p^2 + \frac{m^2}{z^2})y_p y_{-p}\}$$

Scalar field in AdS_{d+1}! END OF STEP 2

Higher dimensions

- Multiply and divide by z^d:

$$S_B = -\int dz z^{-d+1} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \frac{\partial x_p}{\partial z} \frac{\partial x_{-p}}{\partial z} \frac{1}{z^{-d}\dot{G}(p)}$$

- Let $z^{-d}\dot{G} = -f^2$ and require as before:

$$z^{d-1}(z^{-d+1}\frac{d}{dz})^2 e^{-\ln f} = z^{-d+1}(p^2 + \frac{m^2}{z^2})e^{-\ln f}$$

- Performing the same manipulations as before we get:

$$S_B = \int dz \int_p \{z^{-d+1}(\frac{\partial y_p}{\partial z}\frac{\partial y_{-p}}{\partial z}) + z^{-d+1}(p^2 + \frac{m^2}{z^2})y_p y_{-p}\}$$

Scalar field in AdS_{d+1}! END OF STEP 2

Higher dimensions

- Multiply and divide by z^d:

$$S_B = -\int dz z^{-d+1} \int \frac{d^d p}{(2\pi)^d} \frac{1}{2} \frac{\partial x_p}{\partial z} \frac{\partial x_{-p}}{\partial z} \frac{1}{z^{-d} \dot{G}(p)}$$

- Let $z^{-d}\dot{G} = -f^2$ and require as before:

$$z^{d-1}(z^{-d+1}\frac{d}{dz})^2 e^{-\ln f} = z^{-d+1}(p^2 + \frac{m^2}{z^2}) e^{-\ln f}$$

- Performing the same manipulations as before we get:

$$S_B = \int dz \int_p \{z^{-d+1}(\frac{\partial y_p}{\partial z}\frac{\partial y_{-p}}{\partial z}) + z^{-d+1}(p^2 + \frac{m^2}{z^2}) y_p y_{-p}\}$$

Scalar field in AdS_{d+1}! END OF STEP 2

RG evolution

- What have we achieved?

$$\underbrace{e^{-S_I[y_f]}}_{IR\ theory} = \int dy_i \underbrace{\int \mathcal{D}y(z) e^{-\int_{z_i}^{z_f} dz\ S_B[y(z)]}}_{d+1-dimensional\ AdS\ \text{``bulk''}\ theory} \underbrace{e^{-S_I[y_i]}}_{\text{``boundary''}\ UV-theory}$$

- S_I is a perturbation to a d-dimensional CFT. The action S_B depends on the CFT. In our case the CFT is a free field theory. Can be generalized.
- Did not use the AdS/CFT conjecture or string theory - just field theory RG.

RG evolution

- What have we achieved?

$$\underbrace{e^{-S_I[y_f]}}_{IR\ \ theory} = \int dy_i \underbrace{\int \mathcal{D}y(z) e^{-\int_{z_i}^{z_f} dz\ S_B[y(z)]}}_{d+1-dimensional\ \ AdS\ \ \text{``bulk''}\ \ theory} \underbrace{e^{-S_I[y_i]}}_{\text{``boundary''}\ \ UV-theory}$$

- S_I is a perturbation to a d-dimensional CFT. The action S_B depends on the CFT. In our case the CFT is a free field theory. Can be generalized.
- Did not use the AdS/CFT conjecture or string theory - just field theory RG.

RG evolution

- What have we achieved?

$$\underbrace{e^{-S_I[y_f]}}_{IR\ \ theory} = \int dy_i \quad \underbrace{\int \mathcal{D}y(z)e^{-\int_{z_i}^{z_f} dz\ S_B[y(z)]}}_{d+1-dimensional\ \ AdS\ \ \text{``bulk''}\ \ theory} \quad \underbrace{e^{-S_I[y_i]}}_{\text{``boundary''}\ \ UV-theory}$$

- S_I is a perturbation to a *d*-dimensional CFT. The action S_B depends on the CFT. In our case the CFT is a free field theory. Can be generalized.
- Did not use the AdS/CFT conjecture or string theory - just field theory RG.

Example: Green function

- We can work with $x(p)$ - gives exact result. Or with $y(p) \approx e^{-pz} x(p)$ - gives low energy result.
- For eg. take $S_i[x_i] = kx_i$ and evaluate semiclassically to get

$$S_I[x_f] = \frac{1}{2}k^2(G(T) - G(0)) + kx_f$$

- Field theory language: $x_i = \phi = \phi_l + \phi_h, \quad x_f = \phi_l, \quad G(T) - G(0) = \Delta_h, \quad k = J.$ We get the expected Wilson action:

$$S_{I,\Lambda}[\phi_l] = -\frac{1}{2}J\Delta_h J + J\phi_l$$

Example: Green function

- We can work with $x(p)$ - gives exact result. Or with $y(p) \approx e^{-pz}x(p)$ - gives low energy result.
- For eg. take $S_i[x_i] = kx_i$ and evaluate semiclassically to get

$$S_I[x_f] = \frac{1}{2}k^2(G(T) - G(0)) + kx_f$$

- Field theory language: $x_i = \phi = \phi_l + \phi_h, \quad x_f = \phi_l, \quad G(T) - G(0) = \Delta_h, \quad k = J.$ We get the expected Wilson action:

$$S_{I,\Lambda}[\phi_l] = -\frac{1}{2}J\Delta_h J + J\phi_l$$

Example: Green function

- We can work with $x(p)$ - gives exact result. Or with $y(p) \approx e^{-pz} x(p)$ - gives low energy result.
- For eg. take $S_i[x_i] = kx_i$ and evaluate semiclassically to get

$$S_I[x_f] = \frac{1}{2} k^2 (G(T) - G(0)) + kx_f$$

- Field theory language:
  $x_i = \phi = \phi_l + \phi_h, \quad x_f = \phi_l, \quad G(T) - G(0) = \Delta_h, \quad k = J.$
  We get the expected Wilson action:

$$S_{I,\Lambda}[\phi_l] = -\frac{1}{2} J \Delta_h J + J \phi_l$$

Example: Green function

- We can work with $x(p)$ - gives exact result. Or with $y(p) \approx e^{-pz}x(p)$ - gives low energy result.
- For eg. take $S_I[x_i] = kx_i$ and evaluate semiclassically to get

$$S_I[x_f] = \frac{1}{2}k^2(G(T) - G(0)) + kx_f$$

- Field theory language:
$x_i = \phi = \phi_l + \phi_h, \quad x_f = \phi_l, \quad G(T) - G(0) = \Delta_h, \quad k = J.$
We get the expected Wilson action:

$$S_{I,\Lambda}[\phi_l] = -\frac{1}{2}J\Delta_h J + J\phi_l$$

So we have derived the AdS/CFT prescription - but only for for the simplest case - Gaussian theory.

The ERG (in terms of x) has a finite cutoff (but at a fixed point it is conformally invariant). The AdS version in terms of y is a low energy ($p << \Lambda$) "continuum" CFT - which is what is usually studied in the literature.

So we have derived the AdS/CFT prescription - but only for for the simplest case - Gaussian theory.
The ERG (in terms of x) has a finite cutoff (but at a fixed point it is conformally invariant). The AdS version in terms of y is a low energy ($p << \Lambda$) "continuum" CFT - which is what is usually studied in the literature.

Generalization

- Generalize to a non trivial fixed point:

$$S_{Fixed\ Point} = \frac{1}{2}x^2 G^{-1} + S_0(x)$$

$$S = S_{Fixed\ Point} + S_1(x) = \frac{1}{2}x^2 G^{-1} + S_0(x) + S_1(x)$$

- Both S_0 and $S_0 + S_1$ obey Polchinski equation. Taking the difference we get

$$\frac{\partial S_1}{\partial t} = \frac{1}{2}\dot{G}[\underbrace{-\frac{\partial^2 S_1}{\partial x^2} + (\frac{\partial S_1}{\partial x})^2}_{Gaussian\ part} + 2(\frac{\partial S_0}{\partial x})(\frac{\partial S_1}{\partial x})] .$$

Generalization

- Generalize to a non trivial fixed point:

$$S_{Fixed\ Point} = \frac{1}{2}x^2 G^{-1} + S_0(x)$$

$$S = S_{Fixed\ Point} + S_1(x) = \frac{1}{2}x^2 G^{-1} + S_0(x) + S_1(x)$$

- Both S_0 and $S_0 + S_1$ obey Polchinski equation. Taking the difference we get

$$\frac{\partial S_1}{\partial t} = \frac{1}{2}\dot{G}[\underbrace{-\frac{\partial^2 S_1}{\partial x^2} + (\frac{\partial S_1}{\partial x})^2}_{Gaussian\ part} + 2(\frac{\partial S_0}{\partial x})(\frac{\partial S_1}{\partial x})] .$$

Non trivial FP

- It can be shown that this gives the following “bulk" action for the evolution operator:

$$S_B[x(p,t)] = \int dt \int \frac{d^d p}{(2\pi)^d} \, [\frac{1}{\dot{G}(p)}(\frac{dx(p,t)}{dt})(\frac{dx(-p,t)}{dt}) + \underbrace{\dot{G}(p)((\frac{\delta S_0[x(p,t),t]}{\delta x(p,t)})(\frac{\delta S_0[x(p,t),t]}{\delta x(-p,t)}) - \frac{\delta^2 S_0[x(p,t),t]}{\delta x(p)\delta x(-p)})]}_{New\ term} \, .$$

- $S_0[x(p,t),t]$ is a known function of t (i.e. the coupling constants) and contains the information of the fixed point.
- One can change variables to y and obtain a non trivial *AdS* action for a scalar field.

Non trivial FP

- It can be shown that this gives the following "bulk" action for the evolution operator:

$$S_B[x(p,t)] = \int dt \int \frac{d^d p}{(2\pi)^d} \Big[\frac{1}{\dot{G}(p)}\Big(\frac{dx(p,t)}{dt}\Big)\Big(\frac{dx(-p,t)}{dt}\Big) + \underbrace{\dot{G}(p)\Big(\Big(\frac{\delta S_0[x(p,t),t]}{\delta x(p,t)}\Big)\Big(\frac{\delta S_0[x(p,t),t]}{\delta x(-p,t)}\Big) - \frac{\delta^2 S_0[x(p,t),t]}{\delta x(p)\delta x(-p)}\Big)\Big]}_{New\ \ term} .$$

- $S_0[x(p,t),t]$ is a known function of t (i.e. the coupling constants) and contains the information of the fixed point.
- One can change variables to y and obtain a non trivial AdS action for a scalar field.

Non trivial FP

- It can be shown that this gives the following “bulk" action for the evolution operator:

$$S_B[x(p,t)] = \int dt \int \frac{d^dp}{(2\pi)^d} \, [\frac{1}{\dot{G}(p)}(\frac{dx(p,t)}{dt})(\frac{dx(-p,t)}{dt}) + \underbrace{\dot{G}(p)((\frac{\delta S_0[x(p,t),t]}{\delta x(p,t)})(\frac{\delta S_0[x(p,t),t]}{\delta x(-p,t)}) - \frac{\delta^2 S_0[x(p,t),t]}{\delta x(p)\delta x(-p)})]}_{New\ \ term} .$$

- $S_0[x(p,t),t]$ is a known function of t (i.e. the coupling constants) and contains the information of the fixed point.
- One can change variables to y and obtain a non trivial *AdS* action for a scalar field.

Anomalous Dimension

- One of the interesting things about the AdS description is that the two point function has information about the dimension of the operator.

$$\Delta = \frac{D}{2} \pm \underbrace{\sqrt{m^2 + \frac{D^2}{4}}}_{\nu}$$

- Our starting point (Polchinski's ERG) did not have any such parameter.

Anomalous Dimension

- One of the interesting things about the AdS description is that the two point function has information about the dimension of the operator.

$$\Delta = \frac{D}{2} \pm \underbrace{\sqrt{m^2 + \frac{D^2}{4}}}_{\nu}$$

- Our starting point (Polchinski's ERG) did not have any such parameter.

Anomalous Dimension

- Need to modify this equation: Wilson's original equation had such a parameter ($\psi = e^{-S}$):

$$\frac{\partial \psi}{\partial t} = \dot{g}[\frac{\partial}{\partial x(p)}(\frac{\partial}{\partial x(-p)} + x(p))]\psi[x(p), t]$$

$$\dot{g} = 1 - \frac{\eta}{2} + \frac{p^2}{\Lambda^2}$$

- Wilson Actions for interacting theories are fixed point solutions of these modified equations.

Anomalous Dimension

- Need to modify this equation: Wilson's original equation had such a parameter ($\psi = e^{-S}$):

$$\frac{\partial \psi}{\partial t} = \dot{g}[\frac{\partial}{\partial x(p)}(\frac{\partial}{\partial x(-p)} + x(p))]\psi[x(p), t]$$

$$\dot{g} = 1 - \frac{\eta}{2} + \frac{p^2}{\Lambda^2}$$

- Wilson Actions for interacting theories are fixed point solutions of these modified equations.

Concrete Example

- It is difficult to handle Wilson Actions for interacting theories. So consider a Gaussian theory with anomalous dimension.
- $$S_\Lambda[\phi] = \frac{1}{2}\int_p \frac{p^2}{K(p)} \frac{1}{1 + K(p)((\frac{p}{\mu})^\eta - 1)} \phi(p)\phi(-p)$$
- Then one can show that

$$W_\Lambda[J] = \frac{1}{2}\int_p J(p) \frac{1}{p^2(\frac{p}{\mu})^{-\eta} + \underbrace{\frac{p^2 K}{1-K}}_{R(p)}} J(-p)$$

Concrete Example

- It is difficult to handle Wilson Actions for interacting theories. So consider a Gaussian theory with anomalous dimension.
- $$S_\Lambda[\phi] = \frac{1}{2}\int_p \frac{p^2}{K(p)} \frac{1}{1 + K(p)((\frac{p}{\mu})^\eta - 1)} \phi(p)\phi(-p)$$
- Then one can show that

$$W_\Lambda[J] = \frac{1}{2}\int_p J(p) \frac{1}{p^2(\frac{p}{\mu})^{-\eta} + \underbrace{\frac{p^2 K}{1-K}}_{R(p)}} J(-p)$$

Concrete Example

- This is not singular as $p \to 0$ as long as $\Lambda > 0$.
- $$R \to const \quad ; \quad \frac{p}{\Lambda} \to 0$$
  $$R \to 0 \quad ; \frac{p}{\Lambda} \to \infty$$
- So
  $$\lim_{\Lambda \to 0} W_\Lambda[J] = \frac{1}{2} \int_p J(p) \frac{1}{p^2 (\frac{p}{\mu})^{-\eta}} J(-p)$$

Concrete Example

- This is not singular as $p \to 0$ as long as $\Lambda > 0$.
- $$R \to const \quad ; \quad \frac{p}{\Lambda} \to 0$$
  $$R \to 0 \quad ; \frac{p}{\Lambda} \to \infty$$
- So
  $$\lim_{\Lambda \to 0} W_\Lambda[J] = \frac{1}{2}\int_p J(p)\frac{1}{p^2(\frac{p}{\mu})^{-\eta}}J(-p)$$

Concrete Example

- This is not singular as $p \to 0$ as long as $\Lambda > 0$.
- $$R \to const \quad ; \quad \frac{p}{\Lambda} \to 0$$
  $$R \to 0 \quad ; \frac{p}{\Lambda} \to \infty$$
- So
  $$\lim_{\Lambda \to 0} W_\Lambda[J] = \frac{1}{2}\int_p J(p)\frac{1}{p^2(\frac{p}{\mu})^{-\eta}}J(-p)$$

Concrete Example

But for small p

$$S_\Lambda[\phi] = \frac{1}{2}\int_p \frac{p^2}{(\frac{p}{\mu})^\eta}\phi(p)\phi(-p)$$

This is unusual. Wilson action is normally analytic at $p = 0$ because of the IR cutoff.

Could happen with composite fields. Because composite fields start of with unusual dimensions.

Concrete Example

But for small p

$$S_\Lambda[\phi] = \frac{1}{2}\int_p \frac{p^2}{(\frac{p}{\mu})^\eta}\phi(p)\phi(-p)$$

This is unusual. Wilson action is normally analytic at $p=0$ because of the IR cutoff.
Could happen with composite fields. Because composite fields start of with unusual dimensions.

Concrete Example

- $f(z,p)$ is given by

$$\frac{1}{f(z,p)} \approx const \;\; p(\frac{p}{\mu})^{-\frac{\eta}{2}} I_{-\nu}(pz)$$

and

$$G \approx \frac{1}{p^2(\frac{p}{\mu})^{-\eta}\frac{I_{-\nu}(pz)}{I_{\nu}(pz)}}$$

with $\nu = 1 - \frac{\eta}{2}$.

Anomalous Dimension

- Using $I_\nu(pz) \approx (pz)^{-\nu}$ as $pz \to 0$

$$\lim_{pz\to 0} G = constant$$

No singularity at $p = 0$, as it should be because it is the high energy propagator.

- For large pz, $I_\nu(pz) \to e^{pz}$ so

$$\lim_{pz\to\infty} G = \frac{1}{p^2(\frac{p}{\mu})^{-\eta}}$$

as required.

Anomalous Dimension

- How did the field transformation introduce anomalous dimension? The general integrating kernel of such differential equations are of the form:

$$\psi[x_f(p), t_f] = \int dx_i \; e^{-\frac{1}{2}A^2(p,t)[x_f(p) - Z(p,t_f,t_i)x_i(p)]^2} \psi[x_i(p), t_i]$$

with $Z \approx e^{-p^2 + \frac{\eta}{2}}$.

- We started with:

$$\psi'(x_f, t_f) = \int dx_i \; e^{\frac{1}{2}\frac{(x_f - x_i)^2}{G_f - G_i}} \psi'(x_i, t_i)$$

which has $Z = 1$ so no anomalous dimension.

Anomalous Dimension

- How did the field transformation introduce anomalous dimension? The general integrating kernel of such differential equations are of the form:

$$\psi[x_f(p), t_f] = \int dx_i \; e^{-\frac{1}{2}A^2(p,t)[x_f(p) - Z(p,t_f,t_i)x_i(p)]^2} \psi[x_i(p), t_i]$$

with $Z \approx e^{-p^2 + \frac{\eta}{2}}$.

- We started with:

$$\psi'(x_f, t_f) = \int dx_i \; e^{\frac{1}{2}\frac{(x_f - x_i)^2}{G_f - G_i}} \psi'(x_i, t_i)$$

which has $Z = 1$ so no anomalous dimension.

Anomalous Dimension

- How did the field transformation introduce anomalous dimension? The general integrating kernel of such differential equations are of the form:

$$\psi[x_f(p), t_f] = \int dx_i \; e^{-\frac{1}{2}A^2(p,t)[x_f(p) - Z(p,t_f,t_i)x_i(p)]^2} \psi[x_i(p), t_i]$$

with $Z \approx e^{-p^2 + \frac{\eta}{2}}$.

- We started with:

$$\psi'(x_f, t_f) = \int dx_i \; e^{\frac{1}{2}\frac{(x_f - x_i)^2}{G_f - G_i}} \psi'(x_i, t_i)$$

which has $Z = 1$ so no anomalous dimension.

Anomalous Dimension

- But we made a change of variables: $x = f_\nu y$.
- f_ν has the behaviour $e^{(1-\nu)t}$:

$$\psi[y_f(p), t_f] \approx \int dy_i \, e^{-\frac{1}{2}A^2(p,t)[f_\nu(p,t_f)y_f(p) - f_\nu(p,t_i)y_i(p)]^2} \psi[y_i(p), t_i]$$

So $Z(t_f, t_i) \approx \frac{f_\nu(p,t_i)}{f_\nu(p,t_f)} \approx e^{\frac{\eta}{2}(t_f - t_i)}$ and $\nu = 1 - \frac{\eta}{2}$ as expected.

Anomalous Dimension

- But we made a change of variables: $x = f_\nu y$.
- f_ν has the behaviour $e^{(1-\nu)t}$:

$$\psi[y_f(p), t_f] \approx \int dy_i \, e^{-\frac{1}{2}A^2(p,t)[f_\nu(p,t_f)y_f(p) - f_\nu(p,t_i)y_i(p)]^2} \psi[y_i(p), t_i]$$

So $Z(t_f, t_i) \approx \frac{f_\nu(p,t_i)}{f_\nu(p,t_f)} \approx e^{\frac{\eta}{2}(t_f - t_i)}$ and $\nu = 1 - \frac{\eta}{2}$ as expected.

Anomalous Dimension

- So we understand how anomalous dimension gets incorporated. This is relevant for elementary fields in non trivial fixed point theories. As well as composite operators.

Summary

- We have a holographic form of Wilson's exact RG.
- A change of variables maps this to an action in AdS space - makes contact with "AdS/CFT" - without invoking the Maldacena conjecture or string theory.
- This ERG prescription becomes significant when you calculate quantities that are not already determined by conformal symmetry.
- Explicit calculations have been done only for the free theory - Gaussian fixed point - but with anomalous dimensions.

Summary

- We have a holographic form of Wilson's exact RG.
- A change of variables maps this to an action in AdS space - makes contact with "AdS/CFT" - without invoking the Maldacena conjecture or string theory.
- This ERG prescription becomes significant when you calculate quantities that are not already determined by conformal symmetry.
- Explicit calculations have been done only for the free theory - Gaussian fixed point - but with anomalous dimensions.

Summary

- We have a holographic form of Wilson's exact RG.
- A change of variables maps this to an action in AdS space - makes contact with "AdS/CFT" - without invoking the Maldacena conjecture or string theory.
- This ERG prescription becomes significant when you calculate quantities that are not already determined by conformal symmetry.
- Explicit calculations have been done only for the free theory - Gaussian fixed point - but with anomalous dimensions.

Summary

- We have a holographic form of Wilson's exact RG.
- A change of variables maps this to an action in AdS space - makes contact with "AdS/CFT" - without invoking the Maldacena conjecture or string theory.
- This ERG prescription becomes significant when you calculate quantities that are not already determined by conformal symmetry.
- Explicit calculations have been done only for the free theory - Gaussian fixed point - but with anomalous dimensions.

- In principle it can be done for non trivial fixed point. This has not been done yet (but anomalous dimensions have been understood).
- Need to study composite operators.
- Interactions need to be understood.

And most importantly - the significance of dynamical gravity (and especially black holes) in the bulk needs to be studied.

- In principle it can be done for non trivial fixed point. This has not been done yet (but anomalous dimensions have been understood).
- Need to study composite operators.
- Interactions need to be understood.

And most importantly - the significance of dynamical gravity (and especially black holes) in the bulk needs to be studied.

- In principle it can be done for non trivial fixed point. This has not been done yet (but anomalous dimensions have been understood).
- Need to study composite operators.
- Interactions need to be understood.

And most importantly - the significance of dynamical gravity (and especially black holes) in the bulk needs to be studied.

- In principle it can be done for non trivial fixed point. This has not been done yet (but anomalous dimensions have been understood).
- Need to study composite operators.
- Interactions need to be understood.

And most importantly - the significance of dynamical gravity (and especially black holes) in the bulk needs to be studied.

THANK YOU!