On the entropy of supersymmetric black holes in string theory

Black Hole Entropy

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Plan of talks

Introduction to black hole entropy

Black holes in string theory

The microscopic computation

Mathieu moonshine

$L_2(11)$ Moonshine

Concluding Remarks

Introduction to black holes

What is a black hole?

- Consider a spherical planet/star with mass M and radius R. The escape velocity from the object is

$$\frac{1}{2}mv^2 = \frac{G_N M m}{R}$$

- When $R < R_s \equiv 2G_N M/c^2$, one needs $v > c$ – nothing, not even light, can escape from the surface of such an object. The object is **black**. [Michell (1784), Laplace (1798)]

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- In GR, the metric due to a star/planet [Schwarzschild (1916)]

$$ds^2 = \boxed{\left(1 - \frac{R_S}{r}\right)} c^2 dt^2 - \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \,.$$

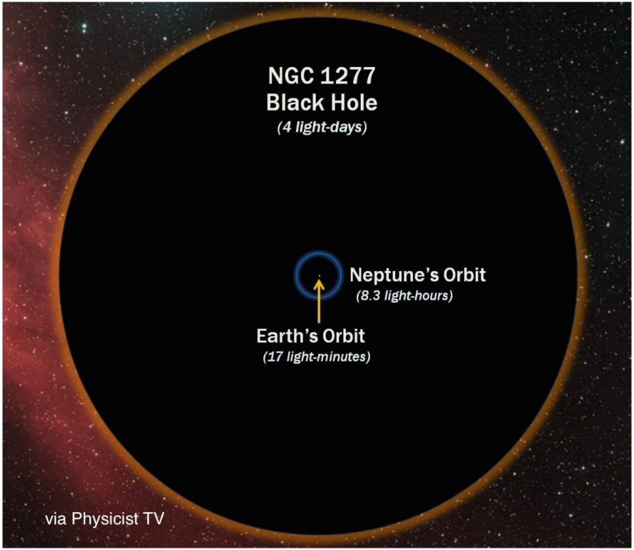
- When the object has a radius less than R_S, then it is a black hole with an event horizon at $r = R_S$.
- The Schwarzschild radius for the sun is about 3 kilometres.

Formation of black holes

1. Stellar collapse: When a star exhausts is source of thermonuclear fuel, it will collapse due to its own gravitational atraction. The end-point depends on its mass M.
   - For $M < 1.4 M_\odot$, the pressure due to electrons provides stabiilty leading to white dwarfs.
   - For $< 1.4 M_\odot < M \lesssim 2 M_\odot$, electrons combine with protons to form neutrons and neutron pressure stabilises leading to neutron stars.
   - For $M > 2 M_\odot$, if the star does not become a supernova, then it will continue to collapse until it becomes a black hole.

   The typical mass range for such black holes is expected to be $2 M_\odot < M \lesssim 100 M_\odot$.
2. The centres of galaxies seem to contain huge black holes with masses in the range $[10^5, 10^{10}] M_\odot$.
3. Primordial black holes: These were formed in the early universe in regions of high density. There is no range for their masses and we have not seen any.

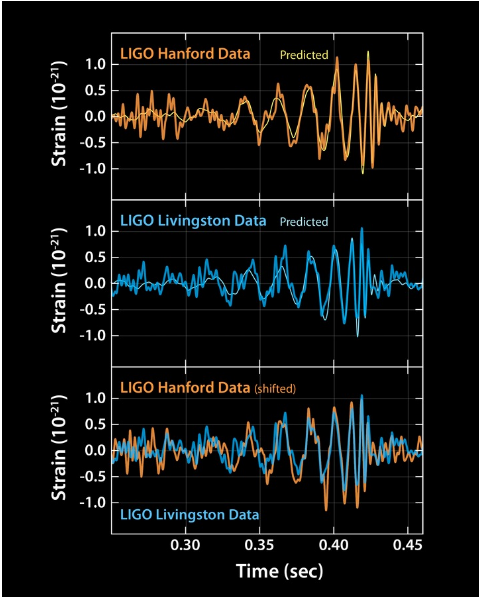
The massive black hole in NGC 1277 2-5$\times 10^9 M_\odot$



GW150914 – the first detection of gravitational waves

The announcement

On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory (LIGO) simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}.



The image of the black hole

The image shows a bright ring formed as light bends in the intense gravity around a black hole that is 6.5 billion times more massive than the Sun. Credit: Event Horizon Telescope Collaboration

Charged black holes

- The Reissner-Nordström black hole is a solution to Einstein-Maxwell's equations. (in units where $c = 1$ and $G_N = 1$)

$$ds^2 = f(r)\ dt^2 - f(r)^{-1}dr^2 - r^2 d\Omega^2\ .$$

with $f(r) = 1 - \frac{2M}{r} + \frac{q_e^2}{r^2}$ and electric field $F_{rt} = \frac{q_e}{4\pi r^2}$.
- It is the most general spherically symmetric static soln.
- There are three situations ($D \equiv (M^2 - q_e^2)$)
  - $[D > 0]$ There are two horizons at $M \pm \sqrt{D}$. non-extremal
  - $[D = 0]$ Both the horizons coincide. extremal
  - $[D < 0]$ There is no horizon. naked singularity
- **Cosmic Censorship Hypothesis:**
  Nature forbids nakedness. [Penrose (1969)]

Black hole thermodynamics

Bekenstein (1972-73) observed that black holes satisfied laws similar to those in thermodynamics. [See Scholarpedia]

- **Zeroth Law:** The surface gravity κ of a black hole is a constant on the horizon. $\kappa = -f'(r_h)/2$
- **First Law:** $\delta M = \frac{\kappa}{8\pi}\,\delta A + \phi\,\delta q_e$. $A = 4\pi r_h^2$
- **Second Law:** $\delta A \geq 0$ in any process. [Hawking]

Bekenstein's analogy suggests the identifications:

$$U = M\ ; \quad T \sim \kappa \ \text{ and } \ S \sim A\ .$$

Using semi-classical arguments, Hawking (1974) showed that black holes behave as black bodies with a temperature, $T_H = \frac{\hbar\kappa}{2\pi}$. This is known today as the Hawking temperature.

Black hole thermodynamics

- Using $dU = TdS$, we see that

$$S = \frac{A}{4\hbar} = k_B \frac{Ac^2}{4G_N\hbar} = \frac{k_B}{\ell_{Pl}^2} \quad \frac{A}{4}$$

- This implies that a black hole carries an entropy proportional to the area measured in Planck units.
- The Hawking temperature of the RN black hole is

$$T_H = \frac{\hbar\sqrt{M^2 - q_e^2}}{2\pi r_h^2} \ , \quad r_h \equiv M + \sqrt{M^2 - q_e^2}$$

- An extremal RN black hole thus has $T_H = 0$.
- In stat. mechanics, at zero temperature, one has

$$S_{\mathrm{stat}} = k_B \log \mathrm{degeneracy} \ .$$

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- In stat. mechanics, at zero temperature, one has

$$S_{\rm stat} = k_B \log \text{degeneracy} \;.$$

- Can one provide a statistical description of S_{BH} for extremal black holes? $S_{BH} \stackrel{?}{=} S_{\rm stat}$

Understanding black hole entropy

- We have seen that the most general static spherical symmetric black hole solution in GR is described by only two parameters, its mass and charge. How does this huge degeneracy $\propto e^{A/4}$ arise?
- As in thermodynamics, we treat mass and charge as macroscopic variables (like the volume of an ideal gas). One needs a microscopic theory (analogous to the statistical mechanics of an ideal gas).
- One needs to 'count' configurations in this microscopic theory satisfying the macroscopic constraints and compute its degeneracy.
- Finally, one needs to compare it with the result on the macroscopic side.

We will consider only extremal black holes in this talk.

Black holes in string theory

Understanding black hole entropy in string theory

- In order to be able to do the comparison, we should be able to first compute the entropy in the two descriptions – one macroscopic and the other microscopic.
- We will focus on examples that arise from string theory. These differ from GR on several fronts.
  - There are more fields – gauge fields, scalar fields ('moduli').
  - The string effective action has an infinite set of higher derivative corrections.
- On the macroscopic side: Does the area law hold in this setting? Does the entropy depend on the moduli?
- On the microscopic side: How do we carry out the counting?

A first attempt

- String theory has two parameters: ℓ_s – the string length and g_s – the string coupling. Perturbative string theory corresponds to $g_s \to 0$.
- Consider a string state at level n – it has degeneracy $d(n) \sim e^{c\sqrt{n}}$ and mass $M \sim \sqrt{n}/\ell_s$. $\Longrightarrow$ $\boxed{S_{\rm stat} \sim \sqrt{n}}$.
- From the string theory effective action, one has $G_N = g_s^2 \ell_s^2$. $\Longrightarrow$ $\boxed{S_{BH} \sim G_N M^2 \sim g_s^2 n}$.
- Clearly the dependence on n don't agree.
- The Schwarzschild radius $R_s \sim G_N M = g_s^2 \sqrt{n} \ell_s$. The string perturbative regime is when $R_s \ll \ell_s$ (or $g_s^2 \sqrt{n} \ll 1$) while the semi-classical gravity is valid when $g_s^2 \sqrt{n} \gg 1$.
- The two results differ in general but agree when $R_s \sim \ell_s$.

How do we compare?

- A key ingredient in making this comparison work is supersymmetry.
- In supersymmetry, there are quantities called ‘indices’ that are protected from several kinds of corrections and one can safely interpolate from the string regime to the gravity regime.
- This is achieved by considering a class of blackholes that are called BPS (after Bogomolnyi-Prasad-Sommerfield) or supersymmetric black holes – they correspond to solutions that preserve a fraction of the supersymmetry – ‘half’, ‘quarter’ and so on in the string theory under consideration.
- Another simplification is to consider zero-temperature i.e., extremal black holes.
- This approach has lead to many of the successful comparisons that I will describe next.

The first successes [1995-1996]

- For the heterotic string on T^6, Sen shows that electrically charged black holes are realised microscopically as the states of the heterotic string and computes S_{stat}. However, $S_{BH} = 0$ but he argued that higher-derivative corrections lead to a non-zero answer with $S_{stat} \propto S_{BH}$.
- In type II string theory on $K3 \times S^1$, Strominger and Vafa show that $S_{\mathrm{stat}} = S_{BH}$ for an extremal five-dimensional black hole in the limit of large charges. The microscopic system consists of $D1/D5$ branes wrapping $K3 \times S^1$.
- In Sen's example, Dijkgraaf, Verlinde^2 (DVV) magically conjured up a modular form (the Igusa cusp form) generates the degeneracy of extremal dyonic black holes and that $S_{\mathrm{stat}} = S_{BH}$ for large charges.
- So this is the first four-dimensional model where the BH entropy was recovered.

The toy model: $\mathcal{N} = 4$ supersymmetric string theory

- We focus on a four-dimensional string theory compactification obtained by compactifying the 10-dimensional heterotic string on a six-torus.
- String dualities relate this model to the type II string theories compactified on $K3 \times T^2$.
- This theory generically has 28 vector fields and thus charged black holes carry a vector of 28 electric/magnetic charges. Let $\mathbf{q}_e/\mathbf{q}_m$ represent these charges.
- Form the following three 'scalars' under T-duality:

$$\mathbf{q}_e \cdot \mathbf{q}_e \quad , \quad \mathbf{q}_e \cdot \mathbf{q}_m \quad , \quad \mathbf{q}_m \cdot \mathbf{q}_m \ .$$

- The Bekenstein-Hawking entropy of these (dyonic) black holes is

$$S_{BH} = \pi \sqrt{\mathbf{q}_e^2 \ \mathbf{q}_m^2 - (\mathbf{q}_m \cdot \mathbf{q}_m)^2} \ .$$

Small black holes

- For electrically charged black holes, $S_{BH} = 0$. Geometrically, the horizon has zero radius in Einstein-Maxwell theory.
- However, string theory has higher derivative corrections that can change the above conclusion. Consider the following R^2 correction:

$$\delta\mathcal{L} = \phi(a, S)\left[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right]$$

- There is a formula due to Wald that incorporates a class of higher derivative corrections. There is also another formalism called the entropy function method due to Sen.
- Sen computed the entropy for extremal dyonic black holes, with charges $(\mathbf{q}_e, \mathbf{q}_m)$, for this correction

$$S_{BHW} = \pi\sqrt{\mathbf{q}_e^2\ \mathbf{q}_m^2 - (\mathbf{q}_e\cdot\mathbf{q}_m)^2} + 64\pi^2\phi\left(\frac{\mathbf{q}_e\cdot\mathbf{q}_m}{\mathbf{q}_m^2}, \frac{\sqrt{\mathbf{q}_e^2\ \mathbf{q}_m^2 - (\mathbf{q}_e\cdot\mathbf{q}_m)^2}}{\mathbf{q}_m^2}\right) + \cdots$$

The entropy of small black holes

- $S_{BH} = 0$ for these black holes.
- We need to carefully take the limit $\mathbf{q}_m \to 0$ in S_{BHW} for dyonic black holes to obtain the entropy.
- Simple scaling arguments using the tree-level effective action show that $S_{BHW} \propto \sqrt{\mathbf{q}_e^2}$.
- A detailed computation for the heterotic string on T^6 gives for large $\mathbf{q}_e^2$
$$S_{BHW} = 4\pi\sqrt{\frac{\mathbf{q}_e^2}{2}}\ .$$
- This is mapped to a heterotic string state in the microscopic side with level $n = \frac{\mathbf{q}_e^2}{2}$ whose degeneracy is known to be $\exp(4\pi\sqrt{n})$ at large n. Hence $S_{\mathrm{stat}} = S_{BHW}$. [Dabholkar (2006)]

The microscopic computation

(Precision counting)

Precision counting

- AL (Alert Listener) would have noticed that all the matching that we have discussed so far occurred in the limit of large charges.
- AL would ask: Can we go away from this limit?
- On the macroscopic side, we have already seen that the sub-leading contribution to dyonic black holes arose from the R^2 correction.
- We will now see how to organise the counting on the microscopic side. Can we then reproduce the sub-leading contribution for a dyonic black hole?
- The answer is yes! In fact, the computation is done via the saddle-point method and it turns out (not always so!) that the function that has to be extremised is identically the entropy function.

Organising the counting

- Using ideas from statistical mechanics, one sees that it is simpler to construct generating functions for the counting problems.

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- Using ideas from statistical mechanics, one sees that it is simpler to construct generating functions for the counting problems.
- Recall that the canonical partition function in statistical mechanics is the weighted sum over configurations of a fixed energy with weight $\exp(-\beta E)$. For fixed E, the coefficient of $\exp(-\beta E)$ gives the number of states with energy E.
- The counting of BPS states is done in a similar manner. Introduce a fugacity (q_i) for every charge (n_i). One then defines

$$\mathcal{Z}(\mathbf{q}) = \sum_{\mathbf{n}\in L} d(\mathbf{n})\ \mathbf{q}^{\mathbf{n}}$$

  where $d(\mathbf{n})$ is the number of BPS states (microstates) in a macrostate with charge vector $\mathbf{n}$. The charge vector $\mathbf{n}$ is valued in a lattice, L, due to charge quantization.

The physical setting

- Consider type II string theory compactified on $K3 \times T^2$. This is dual to heterotic string theory on T^6.
- This is a 4 dimensional theory with $\mathcal{N} = 4$ supersymmetry.
- Electric and magnetic charges in these objects are valued in an even self-dual lattice L of signature $(6, 22)$.
- $SO(6, 22; \mathbb{Z}) \sim \mathrm{Aut}(L)$ is the 'T-duality' group and $PSL(2, \mathbb{Z})$ is the S-duality group of symmetries.
- The electric and magnetic charges, $(\mathbf{q}_e, \mathbf{q}_m)$, each transform as vectors under the T-duality group.
- The quantization of the charges in terms of (continuous) T-duality invariants is such that

  $$\frac{\mathbf{q}_e^2}{2} \in \mathbb{Z} \quad , \quad \mathbf{q}_e \cdot \mathbf{q}_m \in \mathbb{Z} \quad , \quad \frac{\mathbf{q}_m^2}{2} \in \mathbb{Z} \; .$$

  We will indicate these integers, respectively, by (n, ℓ, m).
- The three invariants transform as a triplet under $PSL(2, \mathbb{Z})$.

Organising the counting of BPS states

- Let $d(n)$ denote the microscopic degeneracy of electrically charged $\frac{1}{2}$-BPS states with charge $\mathbf{q}_e^2/2 = n$. Let

$$\frac{16}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} d(n)\ q^n\ ,$$

where $q = \exp(2\pi i\tau)$.

- Similarly, let $D(n, \ell, m)$ denote the microscopic degeneracy of dyonic $\frac{1}{4}$-BPS states with charges (n, ℓ, m). Let [DVV]

$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \sum_{(n,\ell,m)} D(n, \ell, m)\ q^n r^\ell s^m\ ,$$

where $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$, $r = \exp(2\pi i z)$ and $s = \exp(2\pi i\tau')$ and $\Phi_{10}(\mathbf{Z})$ is the weight ten Igusa cusp from of $Sp(2, \mathbb{Z})$.

Refining the generating functions: Mathieu Moonshine

(SG-KGK, Eguchi-Ooguri-Tachikawa, Cheng, Gaberdiel et al, Eguchi-Hikami, SG, Sutapa Samanta)

What is moonshine?

Moonshine is not a well defined term, but everyone in the area recognizes it when they see it. Roughly speaking, it means weird connections between modular forms and sporadic simple groups. It can also be extended to include related areas such as infinite dimensional Lie algebras or complex hyperbolic reflection groups. Also, it should only be applied to things that are weird and special: if there are an infinite number of examples of something, then it is not moonshine. – R. E. Borcherds

- The classic example is a map that relates conjugacy classes, ρ, of the Monster group to modular functions, $J^{\rho}(\tau)$. This was dubbed Monstrous Moonshine by Conway.
- In this talk, we will consider Mathieu moonshine which is a connection between the largest sporadic Mathieu group, M_{24} and different kinds of automorphic forms.
- We also will see a connection between a simple group $L_2(11)$ and Borcherds-Kac-Moody Lie superalgebras.

Mathieu Moonshine

$$
\begin{tikzcd}
 & \eta_\rho(\tau) \arrow[dr] & \\
\rho \arrow[ur] \arrow[rr] \arrow[dr] & & \Phi_k^\rho(\mathbf{Z}) \\
 & \psi_{0,1}^\rho(\tau, z) \arrow[ur] &
\end{tikzcd}
$$

ρ – conjugacy class of M_{24}
$\eta_\rho(\tau)$ – modular form of $SL(2,\mathbb{Z})$,
$\psi_{0,1}^\rho(\tau, z)$ – Jacobi form of weight 0 and index 1,
$\Phi_k^\rho(\mathbf{Z})$ – modular form of $Sp(2,\mathbb{Z})$

Mathieu Moonshine

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\rho \arrow[ur] \arrow[rr] \arrow[dr] & & \Phi_k^\rho(\mathbf{Z}) \\
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\end{tikzcd}$$

ρ – conjugacy class of M_{24}
$\eta^\rho(\tau)$ – generating function of $\frac{1}{2}$-BPS states,
$\psi_{0,1}^\rho(\tau, z)$ – (twined) elliptic genus of $K3$,
$\Phi_k^\rho(\mathbf{Z})$ – generating function of $\frac{1}{4}$-BPS states.

Refining (twining) the generating function

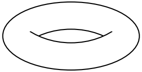
- The generating functions are obtained from computing suitable helicity traces – these are indices in spacetime.
- Let g denote a discrete symmetry of finite order that commutes with supersymmetry.
- Sen refined the index by 'twining' by g. Roughly, one inserts a g into the helicity trace computation.

$$\mathrm{Tr}_{\mathcal{H}}(\cdots) \longrightarrow \mathrm{g}\,\underset{1}{\boxed{}} := \mathrm{Tr}_{\mathcal{H}}(g\ldots)\ .$$

  where $\mathcal{H}$ is a suitable Hilbert space.
- When g is a symplectic automorphism of order N, then the generating function of $\frac{1}{2}$-BPS states (resp. $\frac{1}{4}$-BPS states) turns out to be a modular forms of level N sub-groups of $SL(2,\mathbb{Z})$ (resp. $Sp(2,\mathbb{Z})$). [SG-KG Krishna]

Modular Forms

A modular form $f(\tau)$ of weight k is a function on the upper half plane ($\mathrm{Im}(\tau) > 0$) such that



$$f\big|_k \gamma(\tau) := (c\tau + d)^{-k}\ f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)\ ,$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$. These appear naturally in computing (chiral) partition functions in CFTs on a torus with modular parameter τ.

Example

Consider the contribution of the oscillator part of 24 chiral bosons,

$$\mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\right) = \frac{1}{q\prod_{m=1}^{\infty}(1-q^m)} =: \left[\eta(\tau)^{24}\right]^{-1}\ ,$$

is a modular form of weight 12.

$\eta(\tau)$ however is only a modular form of a sub-group of $SL(2,\mathbb{Z})$.

Mathieu moonshine and multiplicative eta products

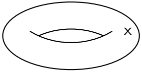
- The heterotic string appears as a $\frac{1}{2}$-BPS soliton in type IIA string theory compactified on $K3$. This identification enables one to explain the appearance of $\eta(\tau)^{-24}$ as the generating function of $\frac{1}{2}$-BPS states in the type IIA string.
- Let $g \in M_{24} \subset S_{24}$ be a discrete symmetry that permutes the 24 chiral bosons with conjugacy class $\rho = 1^{a_1} \cdots N^{a_N}$. Then, the computation after twining by g leads to the map: [SG-KGK]

$$\rho = 1^{a_1} \cdots N^{a_N} \longrightarrow \eta^{\rho}(\tau) := \eta(\tau)^{a_1}\eta(2\tau)^{a_2} \cdots \eta(N\tau)^{a_N} \ .$$

- The cycle shapes (conjugacy classes) of M_{24} are special. They are balanced cycle shapes i.e., there exists a positive integer M such that $\rho = \prod_{i=1}^{N} (\frac{M}{i})^{a_i}$. $\rho = 2^{12}$ has $M = 4$.
- This 'moonshine' for M_{24} was first established by Dummit, Kisilevsky and McKay as well as Mason.

Jacobi Forms

A Jacobi form is a two variable generalisation of a modular form.
The second variable is a point z on the torus.



Under modular transformations,

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z}) \colon\ \tau \to \frac{a\tau+b}{c\tau+d},\ z \to \frac{z}{c\tau+d}.$$

A Jacobi form $g(\tau,z)$ of weight k and index m transforms as

$$g\big|_{k,m}\gamma(\tau,z) := e^{\frac{-2\pi i m c z^2}{c\tau+d}}\ (c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = g(\tau,z)\ ,$$

and under elliptic transf. $z \to z+\lambda\tau+\mu$ (with $(\lambda,\mu)\in\mathbb{Z}^2$)

$$g(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+\lambda z)}\ g(\tau,z)\ .$$

Example

The Jacobi theta functions are Jacobi forms of weight $\frac{1}{2}$ and index $\frac{1}{2}$ of a suitable sub-group of $SL(2,\mathbb{Z})$.

$$\vartheta_1(\tau,z) = \sum_{\ell\in\mathbb{Z}} q^{\frac{1}{2}(\ell+\frac{1}{2})^2}\ r^{(\ell+\frac{1}{2})}\ e^{i\pi\ell}\ ,$$

The elliptic genus is a Jacobi Form

- Let M be a CY manifold of complex dimension d and let $\mathcal{H}(M)$ denote the Hilbert space (in the RR sector) of the two-dimensional $(2,2)$ SCFT of the supersymmetric nonlinear sigma model (nlsm) with target space M.
- The elliptic genus is defined as [Landweber, Schellekens-Warner, Witten]

$$\chi(M;\tau,z) = \mathrm{Tr}_{\mathcal{H}(M)}\Big((-1)^{F_L+F_R} q^{L_0-\frac{d}{8}} \bar{q}^{\bar{L}_0-\frac{d}{8}} e^{2\pi i z J_{0,L}}\Big) \ ,$$

  where $J_{0,L}$ is the $U(1)$ R-charge for left-movers.
- The elliptic genus is a Jacobi form of weight 0 and index $d/2$.

Example (The elliptic genus of K3)

$$\psi_{0,1}(\tau,z) = 8\sum_{j=2}^{4}\left[\frac{\theta_j(\tau,z)}{\theta_j(\tau,0)}\right]^2$$

Mathieu moonshine in the elliptic genus of K3

- Since $K3$ is hyper-Kähler, the supersymmetry of the nlsm is enhanced to $(4,4)$.
- Eguchi and Hikami expanded the elliptic genus in terms of characters, $\mathcal{C}_1$ and $q^{h-\frac{1}{8}}\mathcal{B}_1$, of the $\mathcal{N}=4$ SCA.

$$\psi_{0,1}(\tau,z)=\chi(K3;\tau,z)=24\ \mathcal{C}_1(\tau,z)+q^{-1/8}\ \Sigma(\tau)\ \mathcal{B}_1(\tau,z)$$

- The function $\Sigma(\tau)$ has the following Fourier expansion

$$\Sigma(\tau)=2(-1+45\ q+231\ q^2+770\ q^3+\cdots)$$

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- Eguchi and Hikami expanded the elliptic genus in terms of characters, $\mathcal{C}_1$ and $q^{h-\frac{1}{8}}\mathcal{B}_1$, of the $\mathcal{N}=4$ SCA:

$$\psi^{\rho}_{0,1}(\tau,z) = (1+\chi_{23}(\rho))\,\mathcal{C}_1(\tau,z) + q^{-1/8}\,\Sigma^{\rho}(\tau)\,\mathcal{B}_1(\tau,z)$$

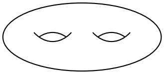
- The function $\Sigma^{\rho}(\tau)$ has the following expansion in terms of characters of M_{24}

$$\Sigma^{\rho}(\tau) = 2(-1+\chi_{45}(\rho)\,q+\chi_{231}(\rho)\,q^2+\chi_{770}(\rho)\,q^3+\cdots)$$

- Eguchi-Ooguri-Tachikawa noticed that the first few numbers are the dimensions of irreps of M_{24}. They conjectured the existence of a family of Jacobi Forms $\psi^{\rho}_{0,1}(\tau,z)$ as given above.
- The conjecture has been proved by Gannon.

Siegel Modular Forms

- Let $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}^2$. It is useful to think of $\mathbf{Z}$ as the period matrix of a genus two surface.



- The $Sp(2,\mathbb{Z})$ matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ acts as $\mathbf{Z} \to M \cdot \mathbf{Z} = (A\mathbf{Z} + B)(C\mathbf{Z} + D)^{-1}$.
- A Siegel modular form of weight k is a function $\Phi_k(Z)$ such that
$$\Phi_k(M \cdot \mathbf{Z}) = \det(CZ + D)^k\ \Phi_k(\mathbf{Z})\ .$$
- Taking $\tau' \to i\infty$ (or $s = e^{2\pi i \tau'} \to 0$), one obtains the Fourier-Jacobi series
$$\Phi_k(\mathbf{Z}) = \sum_{m\geq 0}^{\infty} \phi_{k,m}(\tau, z)\ s^m\ .$$

The Fourier-Jacobi coefficient $\phi_{k,m}(\tau, z)$ is a Jacobi form of weight k and index m.

The Igusa Cusp Form

- The generating function of $\frac{1}{4}$-BPS states is a Siegel moduar form of weight 10. [DVV]
- The Igusa cusp form naturally splits into three parts: [David-Sen]

$$\frac{64}{\Phi_{10}(\mathbf{Z})} = \underbrace{\left[\frac{4\ \eta(\tau)^6}{\theta_1(\tau,z)^2}\right]}_{\text{(i)}} \times \underbrace{\left[\frac{16}{\eta(\tau)^{24}}\right]}_{\text{(ii)}} \times \underbrace{\left[\frac{1}{\mathcal{E}(K3;\mathbf{Z})}\right]}_{\text{(iii)}} ,$$

- Three distinct sectors arise in the type IIB description:
  (i) the overall motion of the $D1$-$D5$ branes in Taub-NUT space.
  (ii) the excitations of the KK-monopole – following the duality chain, these become the states of the heterotic string.
  (iii) the motion of the $D1$-branes in the world volume of the $D5$-branes – this counting leads to the second-quantised elliptic genus (SQEG) of $K3$.
- The SQEG of a Kähler manifold M is [DMVV]

$$\left(s\,\mathcal{E}(M;\mathbf{Z})\right)^{-1} := \left(1 + \sum_{m=1}^{\infty} s^m\ \chi(S^m(M);\tau,z)\right) ,$$

Deconstructing the Igusa Cusp Form

- $S^m(M) = (M^{\times m}/S_m)$ – this is the moduli space of m identical bosonic zero-branes moving on M.
- Let us denote $\chi(S^m(K3);\tau,z)$ by $\psi_{0,m}(\tau,z)$ as it is a Jacobi form of weight zero and index m.
- The CFT for $S^2(K3)$ is constructed as a permutation orbifold.
- Standard CFT arguments show that [DVV]

$$\psi_{0,2}(\tau,z) = \tfrac{1}{2}[\psi_{0,1}(\tau,z)]^2 + T(2)\cdot\psi_{0,1}(\tau,z)\ ,$$

where $T(m)$ is defined as follows:

$$T(m)\cdot\psi_{0,1}(\tau,z) := \frac{1}{m}\sum_{ad=m}\sum_{b=0}^{d-1}\ \psi_{0,1}\left(\tfrac{a\tau+b}{d},az\right)\ .$$

Formulae such as the one above show that $\psi_{0,1}(\tau,z)$ generates all $\psi_{0,m}(\tau,z)$ for $m>1$.

Deconstructing the Igusa Cusp Form

- $S^m(M) = (M^{\times m}/S_m)$ – this is the moduli space of m identical bosonic zero-branes moving on M.
- Let us denote $\chi(S^m(K3); \tau, z)$ by $\psi_{0,m}(\tau, z)$ as it is a Jacobi form of weight zero and index m.
- The CFT for $S^2(K3)$ is constructed as a permutation orbifold.
- CFT arguments show that for twining genera [SG]

$$\psi^{\rho}_{0,2}(\tau, z) = \tfrac{1}{2}[\psi^{\rho}_{0,1}(\tau, z)]^2 + T(2) \cdot \psi^{\rho}_{0,1}(\tau, z) \ ,$$

where $T(m)$ is now re-defined as follows:

$$T(m) \cdot \psi^{\rho}_{0,1}(\tau, z) := \frac{1}{m} \sum_{ad=m} \sum_{b=0}^{d-1} \psi^{\rho_a}_{0,1}\left(\tfrac{a\tau+b}{d}, az\right) \ .$$

where $\rho_a = [g^a]$ when $\rho = [g]$ for some $g \in M_{24}$.

A master formula

- A master formula that subsumes all such formulae is

$$\exp\Big(-\sum_{m=1}^{\infty} s^m\ T(m)\cdot\psi^{\rho}_{0,1}(\tau,z)\Big) = 1 + \sum_{m=1}^{\infty} s^m\ \psi^{\rho}_{0,m}(\tau,z)\ .$$

- This leads to a product formula for the generating function of twined quarter-BPS states.

$$\frac{\Phi^{\rho}_{k}(\mathbf{Z})}{s\ \phi^{\rho}_{k,1}(\tau,z)} = \prod_{\alpha=0}^{N-1}\prod_{m=1}^{\infty}\prod_{n=0}^{\infty}\prod_{\substack{\ell\in\mathbb{Z}\\ 4nm-\ell^2\geq 0}} \Big(1-\omega^{\alpha}q^n r^{\ell} s^m\Big)^{c_{\alpha}(nm,\ell)}$$

  where $c_\alpha(nm,\ell)$ are determined by the coefficients from the Fourier coefficients of $\psi^{\rho_a}_{0,1}(\tau,z)$.
- Are these Siegel modular forms? This product formula is not standard in number theory.

Proposition (SG-KG Krishna, proved by Sutapa Samanta)

To every conjugacy class $\rho = 1^{a_1}2^{a_2}\cdots N^{a_N}$ *of* M_{24}*, there exists a genus two Siegel modular form,* $\Phi_k^\rho(\mathbf{Z})$ *of weight* $k = -2 + \frac{1}{2}\sum_i a_i$ *of a level* N *sub-group of* $Sp(2,\mathbb{Z})$ *such that*

- *its zeroth Fourier-Jacobi coefficient is given by* $\phi^\rho_{k,1}(\tau,z) = \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \times \eta^\rho(\tau)$,
- *and the Jacobi form* $\psi^\rho_{0,1}(\tau,z) = \frac{\phi^\rho_{k,2}(\tau,z)}{\phi^\rho_{k,1}(\tau,z)}$.

There are two standard constructions of Siegel modular forms that work in some but not all cases.

- An additive lift using $\phi^\rho_{k,1}(\tau,z)$:
  $\Phi^\rho_k(\mathbf{Z}) = \mathcal{A}(\phi^\rho_{k,1})(\mathbf{Z})$. [Cléry-Gritsenko]
- A Borcherds product formula:
  $\Phi^\rho_k(\mathbf{Z}) = B_{\psi^\rho_{0,1}}(\mathbf{Z})$. [Cléry-Gritsenko]

$$\begin{tikzcd} & \eta_\rho(\tau) \arrow[dr] & \\ \rho \arrow[ur] \arrow[rr] \arrow[dr] & & \Phi^\rho_k(\mathbf{Z}) \\ & \psi^\rho_{0,1}(\tau,z) \arrow[ur] & \end{tikzcd}$$

The proof

The proof proceeds in two steps. First, there is a natural product formula that one obtains from moonshine. This does not establish modularity. In the second step, following Raum, modularity is proved showing that the product formula is equivalent to the product of multiple rescaled Borcherds products. This is done case by case.

Example (The simplest case)

$$\Phi_{10}(\mathbf{Z}) = \mathcal{A}(\phi_{10,1})(\mathbf{Z}) = B_{\psi_{0,1}}(\mathbf{Z}) \ .$$

Example (where rescaled products are needed)

Let $\rho = [g] = 2^4 4^4$. Then,

$$\left(\Phi_2^{2^4 4^4}(\mathbf{Z})\right)^4 = B_{4\psi^{2^4 4^4}}(\mathbf{Z}) B_{2\psi^{1^8 2^8}}(2\mathbf{Z}) B_{\psi^{1^{24}}}(4\mathbf{Z}) \ .$$

$L_2(11)$ Moonshine

(SG and Sutapa Samanta)

$$\begin{tikzcd}
 & \eta_{\hat{\rho}}(\tau) \arrow[dr] & & \\
\hat{\rho} \arrow[ur] \arrow[rr] \arrow[dr] & & \Delta_k^{\hat{\rho}}(\mathbf{Z}) \arrow[r] & \text{Lie Algebra} \\
 & \psi_{0,1}^{\hat{\rho}}(\tau, z) \arrow[ur] & &
\end{tikzcd}$$

The square-root of the Igusa Cusp Form

- Gritsenko and Nikulin showed that $\Delta_5(\mathbf{Z}) = \sqrt{\Phi_{10}(\mathbf{Z})}$ is associated with the Weyl-Kac-Borcherds (WKB) denominator formula for a Borcherds-Kac-Moody (BKM) Lie superalgebra.
- The Cartan matrix of the real simple roots is

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} .$$

  This is a rank three matrix with one negative eigenvalue.
- This raises a few questions:
  - Does this have anything to do with the physical setting?
  - Does the square-root make sense for all the Siegel modular forms $\Phi_k^\rho(\mathbf{Z})$ that we constructed for all M_{24} conjugacy classes?
  - If yes, is there a BKM Lie superalgebra?

Walls of Marginal Stability $\mathcal{N}=4$ $d=4$ string theory

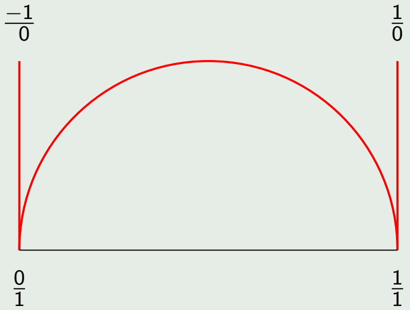
- In $\mathcal{N}=4$ $d=4$ string theory, $\frac{1}{4}$-BPS states can decay into two $\frac{1}{2}$-BPS states as one moves across a wall.
- Let λ denote the complex modulus for the heterotic dilaton-axion field.
- These walls are circular arcs in the upper half-plane given by

$$\left[\mathrm{Re}(\lambda) - \frac{ad+bc}{2ac}\right]^2 + \left[\mathrm{Im}(\lambda) + \frac{\mathcal{E}}{2ac}\right]^2 = \frac{1+\mathcal{E}^2}{4a^2c^2}\,, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$$

  where $\mathcal{E}$ is a real function of all other moduli M. [Sen]
- The arcs intersect the real λ axis at $\frac{b}{a}$ and $\frac{d}{c}$ for any $\mathcal{E}$.
- When $\mathcal{E}=0$, the arcs are semi-circles centred on the real λ-axis with radius $\frac{1}{2ac}$.
- When either $a=0$ or $c=0$, the circles become straight lines perpendicular to the real axis for $\mathcal{E}=0$.

Example (Heterotic string compactified on T^6)

$$\mathcal{F}_1 = \tfrac{-1}{0}, \left(\tfrac{0}{1}, \tfrac{1}{1}\right), \tfrac{1}{0}$$



Cheng and Verlinde showed that these walls of marginal stability get mapped to the walls of the Weyl chamber of the BKM Lie superalgebra

BKM Lie algebras from walls of marginal stability

- Cheng and Verlinde showed that these walls of marginal stability get mapped to the walls of the Weyl chamber of the BKM Lie superalgebra.
- To each edge with vertices $(\frac{b}{a}, \frac{d}{c})$, we can associate a (real) simple root as follows:

$$\left(\tfrac{b}{a}, \tfrac{d}{c}\right) \longleftrightarrow \alpha = \begin{pmatrix} 2bd & ad+bc \\ ad+bc & 2ac \end{pmatrix} \in PGL(2,\mathbb{Z})\ .$$

- The norm of the root is given by $-2\det(\alpha)$.
- The Cartan matrix of the roots can be determined by the inner product induced by the norm. For our example, we obtain

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}\ .$$

- The square-root of the Igusa cusp form provides the automorphic correction to the above KM algebra.

The groups $L_2(11)$, M_{12} and $M_{12} : 2$

- Let $\Omega = PL(11)$ denote the projective line over $\mathbb{F}_{11}$, the field of integers modulo 11.
- Ω be the set of order 12: $(0, 1, 2, 3, \ldots, 9, X = 10, \infty)$.
- Let $\alpha, \beta, \gamma, \delta$ denote the permutations of Ω:

$$\alpha = (\infty)(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X)\ ,\ \ \gamma = (\infty, 0)(1, X), (2, 5), (3, 7)(4, 8)(6, 9)$$
$$\beta = (\infty)(0)(1, 3, 9, 5, 4)(2, 6, 7, X, 8)\ ,\ \ \delta = (\infty)(0)(1)(2, X)(3, 4)(5, 9)(6, 7)(8)$$

- The simple groups $L_2(11)_{A/B}$ and M_{12} are defined as follows:

$$L_2(11)_A = \langle \alpha, \beta, \gamma \rangle; \quad L_2(11)_B = \langle \alpha, \beta, \delta \rangle; \quad M_{12} = \langle \alpha, \beta, \gamma, \delta \rangle.$$

- One has the following sequence of groups

$$L_2(11)_{A/B} \subset M_{12} \xrightarrow{\varphi} M_{12}\!:\!2 \subset M_{24}\ ,$$

where φ represents an outer automorphism of M_{12} given by

$$\alpha^\varphi = \varphi\alpha\varphi^{-1} = \alpha^{-1} \quad , \quad \beta^\varphi = \beta \quad , \quad \gamma^\varphi = \gamma^{-1} \quad , \quad \delta^\varphi = \delta\ .$$

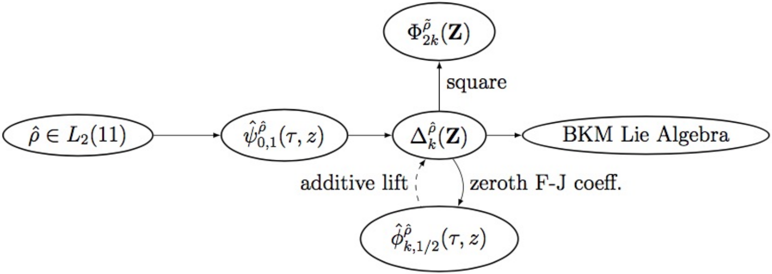
When does the square-root work?

- First, for conjugacy classes of $M_{12}:2$ that map to conjugacy classes of M_{24}, $\psi^{\rho}_{0,1}$ have even coefficients.
- However, $\frac{1}{2}\psi^{\rho}_{0,1}(\tau,z)$ works for all conjugacy classes of $M_{12}:2$ that come from $L_2(11)_{A/B}$ but not for M_{12}.
- For these conjugacy classes, the square-root of the eta product $\eta^{\rho}(\tau)$ also works.
- We can write $\psi^{\hat{\rho}}_{0,1}(\tau,z) = \frac{1}{2}\psi^{\rho}_{0,1}(\tau,z)$ and $\eta^{\hat{\rho}}(\tau) = \sqrt{\eta^{\rho}(\tau)}$.
- For example, $\rho = 2^{12}$ gets related to the $L_2(11)_A$ conjugacy class $\hat{\rho} = 2^6$.
- Let $\Delta^{\hat{\rho}}_{k/2}(\mathbf{Z}) := \sqrt{\Phi^{\rho}_{k}(\mathbf{Z})}$. One can prove that these are also Siegel modular forms.

Proposition (SG and Sutapa Samanta)

Let $\hat{\rho} = 1^{a_1}2^{a_2}\cdots N^{a_N}$ be a conjugacy class of $L_2(11)_A$ or $L_2(11)_B$. The Siegel modular form $\Delta_k^{\hat{\rho}}(\mathbf{Z})$ (with $k = -1 + \frac{1}{2}\sum_i a_i$) provides an automorphic correction to the Kac-Moody Lie algebra with Cartan matrix $A^{(1)}$:

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} .$$



Concluding Remarks

- We have seen that there has been significant progress in the counting of the degeneracy of microstates that contribute to the entropy of black holes.
- The mysterious appearance of the Mathieu group in the considerations enriched the story.
- Can we extend these considerations to string theories with lesser supersymmetry?
- Can we explain the entropy non-extremal black holes? In particular, what can we say about the entropy of the Schwarzschild black hole?